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THE CAMBRIDGE MATHEMATICAL JOURNAL.

VOL. II.

τὰ τοιαῦτα τοῖς μαθηταῖς ἐπὶ σχολῆς φράζουσιν,
οὓς ἂν βούλωνται ὁμοίους αὐτοῖς ποιῆσαι.

PLATO.

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I.—ON THE ELEMENTARY PRINCIPLES OF THE APPLICATION OF ALGEBRAICAL SYMBOLS TO GEOMETRY.

By D. F. GREGORY, B.A. Trin. Coll.

IN several previous papers in this Journal, I have considered the principles on which certain symbols of operation become subject to the same rules of combination as the symbols of number, which are those usually handled in Algebra. The general theory of this subject I gave in a paper (to which I have elsewhere referred) on the Nature of Symbolical Algebra; in which I endeavoured to exhibit distinctly the principles on which various branches of science may be symbolized—that is to say, on which their study is facilitated by expressing the operations by means of symbols. I use the word *operation* for the purpose of avoiding anything like limitation in the subjects which the symbols may represent, as is too apt to be the case when we employ the word *quantity*, which is generally made to be synonymous with number. Among the sciences whose symbolization I there considered, that of Geometry is the most important; and on that account I wish here to treat of it more at large, especially because it appears to me that the theory of the representation of geometrical quantities by numerical symbols is usually but little attended to, and some obscurity still hangs over the subject. In treating of this matter, I may perhaps appear to some to be raising difficulties where there are none; but I think that a little consideration will show to these persons that the question is not quite so simple as might at first be imagined. Much attention has been bestowed on the theory of the representation of

direction by means of the symbols + and —, but the principles on which lines, areas, and solids are represented by numbers has been but little discussed. It is to the latter of these subjects that my remarks will be first directed, and I shall afterwards develop my views of the former.

In the paper I have referred to, I lay down the principle, that an algebraical symbol can only represent an operation in any other science when it is subject to the same laws of combination as that operation. In fact, that as Algebra takes cognizance only of the laws of combination of the symbols, and not of their meaning—in the eye of that science the symbol and the operation are identical. When we turn to the interpretation of our results, we must of course consider the meanings of the symbols—but such interpretation is out of the province of Algebra, and belongs to the science, the operations of which are symbolized. Now, in applying these principles to Geometry, we have first to become acquainted with the operations which require to be symbolized, and then to consider the laws of combination to which they are subject, in order that we may know under which family of algebraical symbols they are to be classed. The ideas with which we are concerned in Geometry are those of magnitude and direction. The former is of three kinds—linear, plane, and solid; and the question is, of what sort of operations these may be considered as the result. Such a one I conceive to be *transference* in *one direction;* for by proper combinations of operations of this description we can represent magnitudes of all kinds. Some persons may think it strange to introduce such an idea as that of transference into so simple a subject as Geometry; but in defence of its adoption, I think it only necessary to plead the simplicity and uniformity of the explanations it affords of the principle of the application of Algebra to Geometry: and I may add, that we are not here considering how Geometry may be treated *geometrically*, but *symbolically;* and we must be content to do so in the way which the subject most readily permits. Besides, for my own part, I think that the idea of transference is quite as simple and elementary as any which occurs in Geometry, and offers itself as readily to the mind of the student. Having fixed on an operation which is to be symbolized, we have also to consider what may be the subject of that operation. The simplest geometrical idea, and that which suits our purpose, is the idea of a point. We may, if we choose, represent this by a symbol, as we represent the fundamental subject-idea in Arithmetic by the symbol 1: but this is not necessary; for, as in Algebra, we have only to consider the combinations of symbols of operation—the subject, being always the same, may be understood, and the symbol for it omitted. Thus it is that we omit in Arithmetic the symbol for unity, which nevertheless requires to be understood at every step as the subject, without which the whole would be unintelligible.

Now, let us assume a to be a symbol representing transference in *one constant* direction through a given space; then, representing

the subject-point by the symbol (.), the compound symbol

$$a(.)$$

will represent a straight line, as the result of transferring a point through a given space in a constant direction. But as we have agreed to omit the subject-symbol, a line of a given length will be simply represented by the symbol a, which now does not represent the operation, but the result of the operation on the subject.

Again, we may combine this symbol with another symbol for transference in some other given direction, and we may ask the meaning of such a combination as

$$b\{a(.)\},$$

or, omitting the symbol for the subject,

$$b(a).$$

This, it is clear, must signify the transference of a line in one constant direction, that is, the line must move parallel to itself, by which means it will trace out a parallelogram, whose sides are represented by a and b.

In the same way in which we have combined two symbols of transference we may combine three, and ask the meaning of the expression $c\{a(b)\}$. This will, on the same principle, represent the transference of a plane in one constant direction, that is, the transference of a plane parallel to itself, the result of which is a parallelopiped. If we combine more symbols than these, we find no geometrical interpretation for the result. In fact, it may be looked on as an impossible geometrical operation; just as $\sqrt{-1}$ is an impossible arithmetical one. For a solid, having equal relations to the three dimensions of space, cannot have any relation with one particular direction, which refers only to one dimension, and direction is essentially involved in the operation we have been considering.

From what has preceded, it appears that we are able, by the combination of the symbol of one kind of operation, to represent the three different geometrical magnitudes—lines, areas, and solids; but, as yet, nothing has been said to point out the *algebraical* nature of these symbols, so that we cannot tell whether or not they coincide algebraically with the symbols for numbers. So far as we have gone, we have not shown how the study of Geometry may be facilitated by having its operations symbolized, as we know not how to treat the symbols, some combinations of which we have been interpreting. But we shall now proceed to show, that these symbols are subject to the two laws of combination which characterize the symbols of number, the ordinary subjects of algebraical operations, viz. the commutative law and the distributive law.

We have found that $b(a)$ represents a parallelogram, the sides of which are a and b; and in the same way $a(b)$ must represent a parallelogram whose sides are also a and b, and which is identical

with the former, as the relative inclination of the sides is the same. Hence it follows, that when a and b represent the geometrical operation of transference in a given direction,

$$a(b) = b(a),$$

or the symbols are *commutative*.

Again, with respect to the distributive law: supposing that the symbol $+$ represents the simple arithmetical idea of addition, (the reason for which restriction will be seen afterwards,) $a + b$ will represent a line resulting from the transference of a point in the same direction through distances a and b, and $c(a + b)$ will represent a parallelogram whose sides are c and $a + b$. But $c(a)$ and $c(b)$ will represent respectively parallelograms, whose sides are c, a and c, b, so that $c(a) + c(b)$ will represent the sum of these parallelograms. But, by the first proposition of the second book of Euclid, we know that the sum of these is equal to the first parallelogram. It is true that the proposition in Euclid is proved only for rectangles, but the principle of the demonstration applies to all parallelograms whatsoever. From this it follows, that when c, a, b represent the geometrical idea of transference in a given direction,

$$c(a + b) = c(a) + c(b),$$

or the symbols are distributive.

We are now enabled to see why we can represent geometrical ideas by arithmetical symbols, so as to render geometrical research easier from our previous acquaintance with arithmetical combinations. It is because the symbols in both cases are subject to the same laws of combination, and therefore in the eye of Algebra are identical, at least so far as these laws (which are the algebraical definitions) are concerned. Whatever, therefore, may have been proved in Arithmetic, in dependence solely on these laws, is equally true in Geometry, provided always that we can interpret the result; for there is no reason why we should always be able to interpret a symbolical result either geometrically or arithmetically. And indeed, in Geometry the uninterpretability is soon presented to us in the combination of more than three symbols of transference. From this it appears why areas and solids may be represented by the product of the symbols of lines, or rather by the apparent product: for when a and b are geometrical symbols, we cannot talk of their being multiplied together—but we see that the operation of one on the other bears a close resemblance to the arithmetical operation of multiplication, and from the identity of the laws of combination they may be considered algebraically as the same, though the meanings be wholly different. This question as to the possibility of representing areas and solids by means of the apparent multiplication of the symbols for lines, has always appeared to me to be one of great difficulty in the application of Algebra to Geometry: nor has the difficulty, I think, been properly met in works on the subject. It is not sufficient to say, as is usually done, that if we divide each of

the lines into a certain number of units, the number of superficial units in the parallelogram will be equal to the product of the number of units in the two lines; it is also necessary to show how a superficial unit can be represented by the product of two linear units, and this I think cannot be done except on the principle which has here been used.

It is to be observed, that in all which has preceded we have supposed the symbols to represent transference in a constant direction. This limitation is necessary in defining our symbols; for if we were to suppose the direction to vary during the progress of the transference, the same laws would not be found to hold with respect to these symbols as we have seen to hold for the symbols we considered, and we should then be unable to reduce geometrical investigations to processes of arithmetical calculation. We might, certainly, if we chose, use symbols representing different kinds of transference, and we might employ ourselves in investigating their nature and the laws of their combination; but having done so, we should derive no assistance from any previous labours in the science of symbols. It is solely from the previous knowledge which we have of the combinations of arithmetical symbols, that we are enabled to facilitate our researches by the application of Algebra to Geometry, or to any science whatever. And thus it is, that any improvement or discovery in Algebra, however isolated and useless it at first appear, may become ultimately of the utmost importance for the prosecution of other branches of knowledge.

Hitherto we have confined ourselves to the consideration of the means of representing symbolically the geometrical ideas of magnitude; and we have shown how the combination of these symbols to represent areas and solids, bears an analogy to the processes of multiplication in Arithmetic: we shall now proceed to consider the symbolization of direction, and to show that the symbols we adopt bear a striking analogy to a well-known arithmetical symbol.

Direction, in ordinary Plane Geometry, is estimated by means of rectilinear angles, which affords us an easy means of symbolizing this geometrical idea; for by supposing a straight line to revolve round a point situate within it, we can make it generate any given angle. This, therefore, is the operation which we shall express by a symbol, and the laws of which we are to investigate. It is clear, in the first place, that if we take some standard angle as that which is to be the result of the operation symbolized, we may produce multiples or submultiples of that angle by performing the operation a certain number of times, or by performing a certain part of the operation. It is therefore necessary to choose some angle for our standard, and the most convenient for our purpose is that produced by a complete revolution of the line, or revolution through four right angles. Let us assume, then, the symbol Λ to represent the operation of making a line revolve through four

right angles, so that, a representing a line in a given direction, $\Lambda(a)$, will represent the same line inclined at an angle equal to four right angles,—that is to say, in a direction coinciding with the original direction. If we repeat the operation, $\Lambda\{\Lambda(a)\}$, or in accordance with ordinary algebraical notation $\Lambda^2(a)$, will represent a line inclined to the original at an angle equal to eight right angles, and so on for any number of times that the operation may be performed. As we have introduced integer indices attached to the operation Λ, we may also use fractional indices, and enquire what is the meaning of such an expression as $\Lambda^{\frac{1}{2}}(a)$ or $\Lambda^{\frac{1}{3}}(a)$. In accordance with the algebraical laws for the combination of indices, we easily see that $\Lambda^{\frac{1}{2}}$ must signify an operation which, being performed twice, will give birth to Λ. Such will be the turning of a line through two right angles, or 180^0, so that $\Lambda^{\frac{1}{2}}(a)$ will represent a line measured in the opposite direction from the original line. In the same way $\Lambda^{\frac{1}{3}}$ must signify the turning of a line through one-third of four right angles, or 120^0, as that operation being performed thrice will be equivalent to the turning of a line through four right angles. And generally $\Lambda^{\frac{1}{n}}$ will signify the turning of a line through the n^{th} part of four right angles, or $\frac{360^0}{n}$. Thus, by the use of the simple algebraical notation of indices, joined to the geometrical operation of turning a line through a given angle, we are able to express the operation of turning a line through any angle whatsoever, and so to express all relations of directions between lines situate in a plane. It is to be observed, that since the operation of turning a line through four right angles, or through any multiple of four right angles, brings it back to its original position, the effect of any number of repetitions of the operation Λ is the same, which may be expressed algebraically by saying that

$$\Lambda^n = \Lambda,$$

n being any integer, which is a law of combination of Λ, and may be considered as its algebraical definition. Now, this is the very law which is known to belong to the arithmetical operation of addition usually represented by $+$, since we have then

$$+ + = +, \text{ and therefore } +^n = +,$$

n being any integer. Hence it appears, that as the arithmetical operation of addition, and the geometrical operation of turning a line through four right angles, are subject to the same law of combination, they are, so far as that is concerned, algebraically identical, and may be represented by the same symbol. Such, indeed, has long been the case, for the arithmetical symbols for addition and subtraction, along with certain modifications of them, are constantly

used to represent geometrical direction. This has given rise to much difficulty and many attempts at explanation; some persons wishing to show that the geometrical operation might be supposed to be derived from the arithmetical, but not finding it very easy to do so in a satisfactory manner—others being inclined to found their views of some points in the arithmetical theory on the basis of the geometrical idea, interpreting the former by the latter. I believe, that the more closely the subject is examined, the more clearly it will be seen, that there is really no resemblance in *kind* between the two operations, but only an identity in the laws of combination; and if this be kept steadily in view, all the difficulties which have been observed in this part of mathematics, and on which so much has been written, will receive a satisfactory explanation. This double meaning of $+$ is the reason of the limitation to the meaning of that symbol assumed in p. 4.

We have only considered the operation Λ or $+$, as we may now term it, in connection with the symbol for a line, as it was with reference to the direction of a line that its definition was made. But this symbol may also receive interpretation in another case, to which its original definition does not directly refer. It is not *necessary* that it should admit of any other geometrical interpretation, but such is found to be the case when it is applied to areas. The position of a line is determined by the direction in which its length lies; but the position of a plane cannot be determined in like manner by its extension, since that has two dimensions, and direction has only one. But the position of an area may be determined by the direction of the face of the plane, which can be referred to that of any straight line inclined to it at a given angle (such as a right angle), so that we know how one plane is related to another if we know in what direction the face of each is presented. Now, supposing an area to revolve round any line in its own plane, we can make it assume any position we please; and it is easy to see that the operation of turning the area completely round is subject to the same law as that of turning the line, that is to say, that when it is repeated any number of times the result is the same, since the area will always present the same face. Hence it follows, that these two operations may be represented by the same symbol; so that if in any process of Analytical Geometry we find the symbol $+$, which was originally applied to the symbol for a line, ultimately applied to the symbol for an area, we are able to interpret it. This view of the meaning of $+$, when applied to the symbol for an area, enables us to offer an explanation of a difficulty in Analytical Geometry.

If x, Λy (fig. 1.) be a system of rectangular coordinates, we know, from what has been previously said concerning the representation of the direction of lines, that any abscissa measured along x will be affected with $+$, and any abscissa measured along x' will be affected with $+^{\frac{1}{2}}$ or $-$; and similarly, any

ordinate measured along Ay will be affected with $+$, and any along Ay' with $+^{\frac{1}{2}}$ or $-$. Therefore the coordinates of a point P will be

$$+x, +y, \quad -x, +y, \quad -x, -y, \quad +x, -y,$$

according as it is in the first, second, third, or fourth quadrant. Now, the rectangle Ax Py being represented by the product of the symbols representing its sides, will be

$$+xy, \quad -xy, \quad +xy, \quad -xy,$$

according as it is in the first, second, third, or fourth quadrant. The question then is, what meaning we are to attach to these expressions. It will be seen by a glance at the figure, that if the rectangle $Ax\,P_1y$ turn round the line Ay, or the line Ax through half a circumference, it will occupy the place of $Ax'\,P_2y$, or $Ay'\,P_4x$, and therefore these rectangles may be considered as resulting from the turning of the original rectangle round Ay or Ax through half a circumference, so as to present the other face of the plane. Now, we have just shown that the operation of turning a plane through a complete circumference, so as to present the same face as before, may be represented by $+$, and therefore the operation of turning it through half a circumference may be represented by $-$. Therefore the negative signs attached to expressions for the rectangles in the second and fourth quadrants, are to be interpreted as signifying that these rectangles are equivalent to the original rectangles turned through half a circumference round Ax or Ay, just as the line Ax' would be produced by turning Ax through half a circumference. With respect to the rectangle in the third quadrant, which forms the chief point of difficulty, it can be derived either from that in the second or that in the fourth, by turning them through half a circumference round Ax' or Ay'. And as both of these rectangles present the face opposite to that of the primary rectangle, it is quite consistent with, and indeed follows from the definition of $+$, that the rectangle in the third segment should be represented by $+xy$, since it is derived from the primary rectangle by that rectangle being turned through a circumference, so that it presents the same face in the same direction as it did at first. Or if we suppose that the area in the second or fourth quadrant, instead of continuing to revolve in the same direction as that by revolving in which it was derived from the area in the first quadrant, revolves back so as to undo the operation previously performed, the same result will follow. For the area in the first quadrant being represented by $+xy$, that in the second, being the former turned round Ay through half a circumference, will be represented by $+^{\frac{1}{2}}+xy$: while the area in the third quadrant, being derived from that in the second by its being turned round Ax' through half a circumference in the opposite direction, will be represented by $+^{-\frac{1}{2}}+^{\frac{1}{2}}+xy$, or $+xy$, as in the first

quadrant, which ought to be the case, as the same face as before is presented. The same result of course will follow, if we consider the area in the third quadrant as derived from that in the fourth.

These explanations of the meanings of the symbols + and —, when applied to areas, are consistent with the original definition, and are closely analogous to their significations when applied to lines, so that I think they must be deemed satisfactory. Should it now be asked whether these principles can be applied to solids, so as to explain the meaning of the symbols + and — prefixed to those of parallelopipeds, I have to answer that they do not; and the reason I conceive to be, as I said before on another subject, that a solid being extended in three dimensions has no relation to one direction, which is essentially only of one dimension. A face or an edge of the solid may be referred to one direction, but the solid itself cannot be so referred. Such expressions as $+abc$ or $-abc$ are, I hold, uninterpretable consistently with the geometrical meaning we attach to the symbols + and —. By calling them uninterpretable, I put them in the same class in Geometry as the symbol $\sqrt{-1}$ in Arithmetic; we do not at present see any interpretation for them, though there is no reason why farther progress and more extended views in Arithmetic and Geometry should not enable us to understand what is at present beyond our comprehension.

II.—ON THE TRANSFORMATION OF A CERTAIN ANALYTICAL EXPRESSION.*

THE following very useful formula, mentioned by the Author of a Paper in the fourth Number, viz.

$$(cy-bz)^2+(az-cx)^2+(bx-ay)^2$$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2,$$

may be considered as a particular case of a more general one; to wit,

$$(m_0n_1-m_1n_0)(m_2n_3-m_3n_2)+(n_0l_1-n_1l_0)(n_2l_3-n_3l_2)$$
$$+(l_0m_1-l_1m_0)(l_2m_3-l_3m_2)$$
$$=(l_0l_2+m_0m_2+n_0n_2)(l_1l_3+m_1m_3+n_1n_3)$$
$$-(l_1l_2+m_1m_2+n_1n_2)(l_0l_3+m_0m_3+n_0n_3),$$

which is easily verified, and may be remembered by observing, that the combinations of indices of the same letter which occur in the positive part of the second member of the equation, are the

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* From a Correspondent.

same as would be found in any positive term of the developement of the first number, and similarly for the negative part. From this formula may be immediately deduced some propositions in Spherical Trigonometry. For let there be any three rectangular axes, and from the origin let any four straight lines be drawn, denoted by the figures 0, 1, 2, 3 respectively; and let $l_0m_0n_0$, $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$, be the cosines of the angles which they make with the positive semiaxes. Also, let six planes be drawn, each containing two of these lines: and let the angle between any two of the lines, 1 and 2 for example, be denoted by the symbol (1.2); and the angle between the plane containing the lines 1, 2, and that containing 0, 3, by the symbol $\begin{pmatrix}1.2\\0.3\end{pmatrix}$. Then we have the following equations, viz.

$$\cos(0.1) = l_0l_1 + m_0m_1 + n_0n_1,$$
$$\cos(2.3) = l_2l_3 + m_2m_3 + n_2n_3,$$
&c.

Moreover, it is easily seen that the cosines of the angles which a perpendicular to the plane 0, 1 makes with the axes, will be the quantities $m_0n_1 - m_1n_0$, $n_0l_1 - n_1l_0$, $l_0m_1 - l_1m_0$, divided respectively by the square root of the sum of their squares, that is, by

$$\{1-(l_0l_1+m_0m_1+n_0n_1)^2\}^{\frac{1}{2}},$$

and so for any other plane.

Hence it is evident, that for the angle between any two planes, for instance, 0, 1 and 2, 3, we shall have

$$\cos\begin{pmatrix}0.1\\2.3\end{pmatrix} = (m_0n_1 - m_1n_0)(m_2n_3 - m_3n_2)$$
$$+ (n_0l_1 - n_1l_0)(n_2l_3 - n_3l_2) + (l_0m_1 - l_1m_0)(l_2m_3 - l_3m_2)$$

divided by

$$\pm\sqrt{\{1-(l_0l_1+m_0m_1+n_0n_1)^2\}\{1-(l_2l_3+m_2m_3+n_2n_3)^2\}};$$

or, substituting for the numerator its value as given by the formula at the beginning of this article,

$$\cos\begin{pmatrix}0.1\\2.3\end{pmatrix} = \pm\frac{\cos(0.2)\cos(1.3) - \cos(0.3)\cos(1.2)}{\sin(0.1).\sin(2.3)}.$$

If we suppose a sphere described about the origin as centre, the four lines we have been considering will meet its surface in four points, which will be the angular points of a quadrilateral figure, whose sides and diagonals will be the intersections of the sphere with the six planes above mentioned. And the formula just written evidently gives the angle between any two opposite sides, or between the two diagonals, in terms of the sides and diagonals, according as we adopt different arrangements of the figures 0, 1, 2, 3.

Thus, let a, b, c, d be the sides, and δ, δ' the diagonals. And

suppose a is opposite to c. Then, if ϕ is the angle between the diagonals, we have

$$\cos\phi = \pm\frac{\cos a . \cos c - \cos b . \cos d}{\sin\delta . \sin\delta'},$$

and calling θ the angle between a and c,

$$\cos\theta = \pm\frac{\cos b \cos d - \cos\delta\cos\delta'}{\sin a . \sin c}.$$

If in this last equation we suppose $d = 0$, the figure becomes a triangle, and the diagonals coincide with the sides a and c respectively; and we get immediately the common formula for the angle opposite the side b, viz.

$$\cos\beta = \frac{\cos b - \cos a\cos c}{\sin a \sin c}.$$

If in the former equation we suppose the diagonals at right angles to one another, we must have $\cos\phi = 0$, and therefore

$$\cos a \cos c = \cos b \cos d.$$

This will include the case of a triangle with a perpendicular drawn from any angle to the opposite sides, if we suppose three of the angular points of the quadrilateral to be in the same great circle. In this case, let a and b be two sides, and α, β the two parts into which the third side is divided by the point where it is met by a perpendicular from the opposite angle, and the equation last written gives evidently

$$\cos a \cos\beta = \cos b \cos\alpha,$$

α being supposed contiguous to a, and β to b.

M. N. N.

III.—ON THE NUMBER OF NORMALS THAT CAN BE DRAWN FROM A GIVEN POINT TO AN ALGEBRAICAL SURFACE: BY M. TERQUEM.

[From Liouville's *Journal de Mathématiques*, vol. IV. p. 175.]

Theorem. The number of normals which can be drawn from a given point to an algebraical surface of the degree m, is equal to $m^3 - m^2 + m$.

Let

$$z^m + z^{m-1}\rho_1 + z^{m-2}\rho_2 + \ldots + z\rho_{m-1} + \rho_m = 0 \ldots\ldots (1)$$

be the equation of an algebraical surface referred to rectangular axes x, y, z; ρ_n being an integer function of xy of the degree n

Taking the given point as the origin, the equations to a normal passing through it are

$$\left.\begin{aligned} x + z\frac{dz}{dx} = 0 \\ y + z\frac{dz}{dy} = 0 \end{aligned}\right\} \dots\dots\dots (2).$$

These equations, each of the degree m, combined with equation (1), which is of the same degree, will give rise to a final equation, the degree of which cannot exceed m^3; thus the number of the normals cannot exceed m^3: but this equation involves besides $m^2 - m$ roots which are foreign to the question. To prove this, let us differentiate equation (1) successively with respect to x and z, and with respect to y and z, when we obtain

$$Qdz + Rdx = 0,$$
$$Qdz + Sdy = 0;$$

where Q, R, S are integer functions of x, y, z: from this equations (2) become

$$\left.\begin{aligned} Qx = Rz \\ Qy = Sz \end{aligned}\right\} \dots\dots\dots\dots (3).$$

It is easy to see that if we make $z = 0$, the function Q is reduced to ρ_{m-1}, and the first member of equation (1) to ρ_m; therefore, taking the equations

$$\begin{aligned} z &= 0, \\ \rho_{m-1} &= 0, \\ \rho_m &= 0, \end{aligned}$$

we deduce from them $m(m-1)$ values of x and y, which satisfy the three equations (1) and (3). These values correspond to points of the surface situate in the plane xy, and lines drawn through these points parallel to the axis of z are tangents to the surface. But these points are foreign to the question; therefore the number of normals is $m^3 - m^2 + m$.

Observation 1. The equation of the degree m^3 resulting from the elimination between the three equations (1) and (3), besides the roots foreign to the question, may involve imaginary roots, which in particular positions of the point will still farther reduce the number of possible normals. When the given point is a centre of curvature the equation involves equal roots.

Observation 2. The number of normals to a given surface which can be drawn through a given point, added to the number of tangents which can be drawn to the algebraical curve of the same degree, is always equal to the cube of the degree.

Observation 3. In algebraical curves the final equation is not encumbered with roots foreign to the question, so that the number of normals which can be drawn through a given point to a curve of the degree m, is m^2.

On a Property of Surfaces of the Second Degree: by M. Terquem.

[From Liouville's *Journal de Mathématiques*, vol. IV. p. 241.]

Theorem. In a surface of the second degree the geometrical locus of the points for which the sum of the squares of the normals to the surface is a constant quantity, is a surface of the second degree concentric with the given surface, and having the direction of its principal axes the same.

Let the equation to the given surface referred to its centre be

$$Ax^2 + A'y^2 + A''z^2 + E = 0 \ldots\ldots (1).$$

Supposing the axes to be rectangular, let a, b, c be the coordinates of a point, and x', y', z' the coordinates of the point where a normal, passing through the point a, b, c, meets the surface. Then, from the equations to the normal we have the relations

$$\frac{x'-a}{Ax'} = \frac{y'-b}{A'y'} = \frac{z'-c}{A''z'} \ldots\ldots (2).$$

By eliminating $y'z'$, $x'z'$, $x'y'$, successively between (1) and (2), we obtain

$$x'^6 + 2pax'^5 + x'^4(nc^2 + mb^2 + q) + \&c. = 0,$$
$$y'^6 + 2p'by'^5 + y'^4(n'c^2 + m'a^2 + q') + \&c. = 0,$$
$$z'^6 + 2p''cz'^5 + z'^4(n''b^2 + m''a^2 + q'') + \&c. = 0,$$

where

$$p = \frac{A'}{A-A'} + \frac{A''}{A-A''};\quad p' = \frac{A}{A'-A} + \frac{A''}{A'-A''};$$
$$p'' = \frac{A}{A''-A} + \frac{A'}{A''-A'};$$
$$m'' = \frac{AA''}{(A-A'')^2};\quad m = m' = \frac{AA'}{(A-A')^2};$$
$$n = \frac{AA''}{(A-A')^2};\quad n' = n'' = \frac{A'A'}{(A'-A'')^2};$$
$$q = \frac{E}{A};\quad q' = \frac{E}{A'};\quad q'' = \frac{E}{A''}.$$

The sum of the six values of

$$z'^2 = 4p''^2c^2 - 2n''b^2 - 2m''a^2 - 2q'',$$
$$y'^2 = 4p'^2b^2 - 2n'c^2 - 2m'a^2 - 2q',$$
$$x'^2 = 4p^2a^2 - 2nc^2 - 2mb^2 - 2q;$$

and the sum of the six values of

$$-(2ax' + 2by' + 2bz') = 4a^2p + 4b^2p' + 4c^2p''.$$

Now, the square of the normal is

$$(x'-a)^2 + (y'-b)^2 + (z'-c)^2;$$

and representing the sum of the six values of this square by constant $2R^2$, we obtain after reduction,

$$a^2[2p^2+2p-(m'+m'')+3]+b^2[2p'^2+2p'-(m+n'')+3]$$
$$+c^2[2p''^2+2p'-(n+n')+3]=R^2+q+q'+q''\ldots\ldots(3).$$

Example. If we take the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

equation (3) becomes

$$\left.\begin{array}{l}\dfrac{x^2}{a^4}\,[a^4+b^4+c^4-a^2(b^2+c^2)] \\ +\dfrac{y^2}{b^4}\,[a^4+b^4+c^4-b^2(a^2+c^2)] \\ +\dfrac{z^2}{c^4}\,[a^4+b^4+c^4-c^2(a^2+b^2)]\end{array}\right\}=R^2-(a^2+b^2+c^2);$$

The same method may be followed for surfaces without a centre.

Observation. In lines of the second degree, the geometrical locus of the points for which the sum of the squares of the normal is constant is a similar line, similarly placed.

IV.—ON THE SYMMETRICAL FORM OF THE EQUATION TO THE PARABOLA.

When the parabola is referred to a diameter and the tangent at its vertex, although the equation then assumes the simplest form, yet as these lines are not symmetrical with respect to the curve, the equation itself is not symmetrical with respect to the variables. In order, therefore, to get the equation under a symmetrical form, we must refer the curve to lines similarly situated with respect to it: such are two tangents to the parabola. If we take them as axes, and their intersection as origin, the equation to the curve assumes a form which bears a curious analogy to the symmetrical equations of the other conic sections and of the straight line, and is sufficiently remarkable in itself to deserve attention.

The general equation to a curve of the second degree is

$$(1)\ldots Ay^2+Bxy+Cx^2+Dy+Ex+F=0;$$

the condition that this should represent a parabola is

$$(2)\ldots B^2=4AC \text{ or } B=\pm 2\sqrt{AC},$$

so that (1) is reduced to

$$(3)\ldots(\sqrt{A}y\pm\sqrt{C}x)^2+Dy+Ex+F=0.$$

Now, let the parabola be referred to the two tangents AB, AC (fig. 2.) as axes, and let $AB = a$, $AC = b$, AB being the axis of x, AC of y. Then, since AB is a tangent at B, if we make $y = 0$ in equation (3), the two corresponding values of x must be each equal to a. In this case equation (3) becomes

$$(4) \ldots \; Cx^2 + Ex + F = 0;$$

and the condition for its being a complete square is

$$(5) \ldots \; E^2 = 4CF;$$

and as each root is equal to a, we have

$$\frac{F}{C} = a^2, \quad \frac{E}{2C} = -a;$$

$$\text{therefore } C = \frac{F}{a^2}, \quad E = -\frac{2F}{a}.$$

In a similar manner we should find that

$$A = \frac{F}{b^2}, \quad D = -\frac{2F}{b},$$

so that equation (3) takes the form

$$(6) \ldots \; \left(\frac{y}{b} \pm \frac{x}{a}\right)^2 - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = 0.$$

If we take the superior sign in the first term, this equation is equivalent to

$$\left(\frac{y}{b} + \frac{x}{a} - 1\right)^2 = 0,$$

which is the equation to a straight line, or rather to two which coincide; we must therefore take the inferior sign, in order that the equation may represent a parabola. If, now, we add $\frac{4xy}{ab}$ to both sides, the equation becomes

$$(7) \ldots \; \left(\frac{y}{b} + \frac{x}{a}\right)^2 - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = \frac{4xy}{ab},$$

the first side of which is a complete square. Extracting, then, the square root on both sides, we have

$$\frac{y}{b} + \frac{x}{a} - 1 = \pm 2\sqrt{\frac{xy}{ab}};$$

or transposing,

$$\frac{y}{b} \mp 2\sqrt{\frac{xy}{ab}} + \frac{x}{a} = 1;$$

the first side of which is also a complete square. Extracting the root again, we finally obtain

$$(8) \ldots \; \sqrt{\frac{y}{b}} \pm \sqrt{\frac{x}{a}} = \pm 1,$$

which is the required symmetrical form.

The form of this equation shows at once, that the curve lies wholly between the positive axes, as neither x nor y can ever become negative. So long as $x<a$ and $y<b$, the positive signs only on both sides must be taken, as the difference between two fractions can never be unity. If $x>a$ and $y>b$, the negative sign only on the left-hand side must be taken, as the sum of two quantities greater than unity can never be equal to unity; and either sign on the right-hand side, according to the relative magnitude of the terms on the left-hand side. If $y>b$ and $x<a$, the negative sign on the first side and the positive on the second are to be taken; and if $x>a$ and $y<b$, the negative sign on both sides. This apparent discontinuity, which renders it necessary to take sometimes one sign and sometimes another, arises from the equation (8) not being the complete form of the equation to the curve. All the cases are included in the expanded form of (9)

$$\frac{y^2}{b^2}-\frac{2xy}{ab}+\frac{x^2}{a^2}-2\left(\frac{y}{b}+\frac{x}{a}\right)+1=0.$$

If we transpose one term of (8) and square both sides, we have

$$\frac{y}{b}=1\pm 2\sqrt{\frac{x}{a}}+\frac{x}{a},$$

$$\text{or } \frac{y}{b}-\frac{x}{a}=1\pm 2\sqrt{\frac{x}{a}},$$

so that $\frac{y}{b}-\frac{x}{a}=1$ is the equation to a diameter passing through C, and similarly

$$\frac{y}{b}-\frac{x}{a}=-1$$

is the equation to a diameter passing through B, and

$$\frac{y}{b}-\frac{x}{a}=0$$

to one passing through A.

This form of the equation affords an easy proof of a problem in the Senate-House Papers for 1833. The enunciation is as follows: If there are three tangents to a parabola, the triangle formed by their intersection is half of that whose angular points are the points of contact.

Let ARS, BPC (fig. 2.) be the triangles; then, taking the equation to the parabola referred to AB, AC as axes, the equation to the tangent is

$$\frac{y}{\sqrt{by_1}}+\frac{x}{\sqrt{ax_1}}=1,$$

where x_1y_1 are the coordinates of the point P.

In this equation, making successively $x=0$, $y=0$, we find

$$\text{AS}=\sqrt{by_1},\quad \text{AR}=\sqrt{ax_1}.$$

Now area $\text{ASR} = \frac{1}{2}\,\text{AR} \,.\, \text{AS} \sin \text{A} = \frac{1}{2}\sqrt{abx_1y_1} \sin \text{C},$
and area $\text{CPB} = \text{ACB} - \text{NPC} - \text{MPB} - \text{AMPN}.$
Now $\text{ACB} = \frac{1}{2}\, ab \sin \text{C},$

$$\text{NPC} = \tfrac{1}{2}\,\text{NC} \,.\, \text{PN} \sin \text{C} = \tfrac{1}{2}\, x_1 (b - y_1) \sin \text{C},$$

$$\text{MPB} = \tfrac{1}{2}\,\text{MB} \,.\, \text{PM} \sin \text{C} = \tfrac{1}{2}\, y_1 (a - x_1) \sin \text{C},$$

and $\text{AMPN} = x_1y_1 \sin \text{C}.$

Hence area $\text{CPB} = \frac{1}{2} \sin \text{C} \{ab - x_1 (b - y_1) - y_1 (a - x_1) - 2x_1y_1\}$

$$= \tfrac{1}{2} \sin \text{C} (ab - bx_1 - ay_1).$$

But, since x_1y_1 are coordinates of a point in the parabola,

$$\sqrt{\frac{x_1}{a}} + \sqrt{\frac{y_1}{b}} = 1,$$

and therefore

$$\frac{x_1}{a} + 2\sqrt{\frac{x_1y_1}{ab}} + \frac{y_1}{b} = 1;$$

and multiplying by ab, and transposing,

$$2\sqrt{ab\, x_1y_1} = ab - bx_1 - ay_1;$$

so that

$$\text{area CPB} = \tfrac{1}{2} \sin \text{C} \,.\, 2\sqrt{abx_1y_1} = \sin \text{C} \sqrt{abx_1y_1}$$

and therefore $\text{ASR} = \frac{1}{2}\,\text{CPB}.$

Since $\text{AR} = \sqrt{ax_1}$ and $\text{AS} = \sqrt{by_1}$, we have, making

$$\text{AR} = x', \quad \text{AS} = y',$$

$$\frac{x'}{a} + \frac{y'}{b} = \frac{\sqrt{ax_1}}{a} + \frac{\sqrt{by_1}}{b} = \sqrt{\frac{x_1}{a}} + \sqrt{\frac{y_1}{b}} = 1 \quad \ldots\ldots(8);$$

and as the equation to BC is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

x' and y' are coordinates of the line BC; so that if from any point Q in BC we draw QS, QR parallel to the axes, the line joining the points where they cut the axes will be a tangent to the parabola. This gives the means of describing a parabola by the ultimate intersection of a line subject to move under a certain condition. For if

$$\frac{x}{m} + \frac{y}{n} = 1$$

be the equation to RS, m and n are subject to the condition

$$\frac{m}{a} + \frac{n}{b} = 1.$$

G.

V.—ON THE CONDITION OF EQUILIBRIUM OF A SYSTEM OF MUTUALLY ATTRACTIVE FLUID PARTICLES.

The generally received theory of the Equilibrium of Fluids, (due in its present form to Euler,) assigns one condition as necessary and sufficient in every case. Mr. Ivory conceives, that when a fluid is acted on by forces arising from the mutual attraction of its particles, a second condition is requisite for equilibrium, and has developed the considerations which have led him to this result, in several papers published in the *Phil. Trans.*, and also in the *Phil. Mag.* The authority of Mr. Ivory on any point of mathematical physics is very great: his decision on one to which he has long directed his attention, would be almost final, were it not opposed to the views of Euler, Laplace, and Poisson. The object of this paper is, to examine how far Mr. Ivory, in a paper published in the *Phil. Mag.* vol. XIII., p. 321, has demonstrated the necessity of the subsidiary condition in question. The writer feels it unnecessary to express the diffidence with which he attempts to consider so difficult a subject; he regrets also his inability to discuss Mr. Ivory's views more at large than the present limits would permit.

In the paper just mentioned, Mr. Ivory states the principal steps of the investigation by which Clairaut was led to the condition of equilibrium of a fluid acted on by forces directed to fixed centres; and proceeds to consider the modifications required to adapt the method to the case of a fluid whose particles are mutually attractive. Clairaut first supposes a mass of fluid in equilibrium, and conceives an infinitesimal stratum added to it, which shall produce equable pressure over the whole surface;—the equilibrium of the original mass A will not be disturbed, and the increased mass $A + \delta A$ will be in equilibrio, when the forces acting on its surface are normal to it. This principle, that forces acting on a free surface must be normal to it, was laid down by Huygens, and is confessedly true. By a repetition of this process, the original mass can be enlarged to any extent; and the condition that the nucleus must be in equilibrio becomes, Mr. Ivory observes, unnecessary, by conceiving it diminished *sine limite*. The mathematical condition of equilibrium is, therefore, the expression of the possibility of adding a stratum which shall produce equable pressure, and at the free surface of which the forces shall be normal to it.

Let us endeavour to put this symbolically. Let the force at the original free surface be F; at the point x, y, z produce the normal, and take a length on it $= \frac{\omega}{F}$, ω being infinitesimal: thus we get a stratum producing an equal pressure ω.

Let $f(x, y, z) = c$ be the equation of the free surface; then F being a function of (x, y, z), all that is requisite for the force at any point of the new free surface to be normal is, that

$$f(x, y, z) = c + \delta c$$

shall be its equation.

Let $$V = \sqrt{(f'x)^2 + (f'y)^2 + (f'z)^2}; \text{ then}$$

$$\delta x = \frac{\omega}{F}\frac{f'x}{V}, \quad \delta y = \frac{\omega}{F}\frac{f'y}{V}, \quad \delta z = \frac{\omega}{F}\frac{f'z}{V} \ldots\ldots (1),$$

and $f(x, y, z) = c = f(x', y', z') - [f'x\,\delta x + f'y\,\delta y + f'z\,\delta z]$ (where $x' = x + \delta x$) by Taylor's theorem;

therefore $$c = f(x', y', z') - \frac{\omega}{F}\frac{1}{V}[(f'x)^2 + (f'y)^2 + (f'z)^2]$$

$$= f(x', y', z') - \omega\frac{V}{F};$$

$$\text{therefore } c = c + \delta c - \omega\frac{V}{F};$$

or $\frac{V}{F}$ = a constant, which we may take for unity; therefore $F = V$. Resolving this force along the axes,

$$X = V\frac{f'x}{V}; \text{ whence } X = f'x, \text{ and so } Y = f'y, \ Z = f'z.$$

ω is the increment of pressure $= \delta p$; multiplying the three equations (1) by X, Y, Z, and adding, we get

$$X\delta x + Y\delta y + Z\delta z = \delta p\frac{V^2}{F\,.\,V};$$

or putting d for δ,

$$dp = Xdx + Ydy + Zdz \ldots\ldots (2),$$

the equation of equilibrium of an homogeneous and incompressible fluid, whose density is unity.

An objector to Clairaut's reasoning might urge, that this result, though certainly sufficient, was not shown to be necessary: he might argue, that a way has been shown of building up a fluid mass; but that it has not been proved that every fluid mass is capable of resolution into the smaller masses, by means of which alone Clairaut investigates the conditions of equilibrium. Unless it be made a direct postulate, that every fluid mass in equilibrio will continue in equilibrio, when the part of it contained between the free surface and any level surface is removed, it is difficult to see how this objection can be met, except by showing that the property assigned by Huygens to a free surface, viz. that the force is normal to it, belongs to every surface of equal pressure, and that consequently Clairaut's reasoning is in reality independent of any construction or resolution of a fluid mass into successive

strata. When we assert, with Clairaut, that a fluid mass in equilibrium is not disturbed by the addition of a stratum producing equal pressure, we imply that the reaction produced at any point of the surface of A, by the pressures exerted over the rest of the surface, *i.e.* the effect of the transmitted pressures, is normal to it. For we know that the forces at the surface are so; and unless the inference first stated is correct, there could be no equilibrium. It hence appears, that Clairaut's axiom is equivalent to this—Equable pressure produces a reaction normal to the surface on which it is applied. But if the force at a surface of equal pressure were not normal to it, there could be no equilibrium, because it is only by the transmitted pressures that it can be established.

Clairaut, as his views are represented by Mr. Ivory, says nothing of the transmission of pressure; but it is impossible to investigate fluid equilibrium without tacit or expressed reference to some distinctive character of fluidity; and in the principle he makes use of, the idea of the transmission of pressure is essential. It appears, then, that the force at a surface of equal pressure is normal to it; and this conclusion is little else than a different way of putting the principles employed by Clairaut. We are now enabled to dispense with any process of constructing a fluid mass.

On referring to the mathematical reasoning employed above, we shall easily see that, substituting two infinitesimally near surfaces of equal pressure for the consecutive free surfaces of Clairaut, the result we arrive at is simply the symbolical expression of the principle just laid down, viz. that the force at a surface of equal pressure is normal to it. A very little attention will show, that (2) is true in every case of fluid equilibrium, and that it is completely equivalent to the principle which it represents. In translating, so to speak, his fundamental idea from the infinitesimal to the fluxionary conception, that namely of successive generation, Clairaut has tacitly introduced a new condition, namely, that a surface of equal pressure will necessarily be a free surface of equilibrium, the superincumbent part being removed.

Mr. Ivory remarks—"The investigation of Clairaut is clear and definite. It evidently assumes that there is no cause tending to disturb the equilibrium of A, except the action of the forces at the surface of A upon the matter of δA. On this account his method fails when there is a mutual attraction between the mass A and the stratum δA. If the mass A attract the matter of the stratum δA, and cause it to press, it follows necessarily that the matter of δA will react, and by its attraction will urge the particles of A to move from their places. In this case, therefore, the equilibrium of A is disturbed by a force which Clairaut has not attended to; and unless the effect of this new force is counteracted, the body of fluid $A + \delta A$ will not be in equilibrium. The principle of the method suggests a remedy for this omission, for it is easy to prove that the

equilibrium of A will not be disturbed by the attraction of the stratum δA, if the resultant of that attraction on every particle in the surface of A be directed perpendicularly to it."

This reasoning satisfactorily shews, that if a fluid mass of attractive matter be increased by a stratum producing equal pressure over the free surface, the equilibrium will be destroyed unless a certain condition is fulfilled, of which the symbolical expression is

$$c = \int [Pdx + Qdy + Rdz],$$

P, Q, R being the attractions, parallel to the axes of coordinates, of an element of that part of a fluid mass which is external to a given level surface. But the necessity of this condition cannot be proved, unless it is shewn to be impossible in any way to increase the mass A, without destroying the equilibrium, supposing it not fulfilled. All that has been shown is, that the mass cannot be increased by attraction producing equable pressure over the free surface. Now, generally speaking, the mass so increased will not fulfil the condition of having the forces at the new free surface normal to it, those acting at the original free surface being of course so. We cannot, therefore, affirm that we have fallen on a case in which the ordinary condition is fulfilled, without producing equilibrium. If, however, we dispense with the limitation, that the stratum added shall produce equable pressure, we lose the simplicity of Clairaut's method, nor can we make any use of his principle, except by setting aside the construction he employs, which confines him to the particular case in which a surface of equal pressure is potentially a free surface.

This has already been done, and the result is the general equation of equilibrium. It remains to show, that it is in all cases sufficient. It is admitted to be sufficient in the case of a fluid acted on by forces tending to fixed centres. We shall endeavour to reduce the general case to this. Conceive a body acted on by a force directed to a fixed point. It may be so placed, as to remain at rest under the action of the force, that is, the resultant of the force upon it is equal to zero. In this position of the body, the centre of force is some point within it. Let the body, remaining in the same position, diminish *sine limite*, being always similar to itself, the resultant of the force upon it is always equal to zero; and ultimately, when the body becomes a physical point, it coincides in position with the centre of force, and is in the same state with respect to the action of other forces upon it, as if this force did not exist.

This being granted, conceive a homogeneous mass of fluid composed of mutually attractive particles, the free surface of which fulfils the required equation

$$Xdx + Ydy + Zdz = 0.$$

Let the attractive power of each particle be conceived transferred

to a fixed centre of force coinciding with it. Then the action of all the other particles on one particle is precisely replaced by that of the fixed centres; and it has been shown, that the resultant of the action of the centre coinciding with a particle on that particle, equals zero. Hence, the supposition we have made does not change, in any way, the forces acting on any particle of the mass. Were the system in its present and former state respectively to move, the motions would be widely different; but in the arbitrary position we have placed it in, the action on it is precisely the same in the two cases. Now, the single equation given above assures its equilibrium, when we regard it as a system acted on by forces directed to fixed centres; and as the hypothesis by which we are enabled to look upon it in this way nowise affects the forces acting on it, it follows, that the system considered as acted on by mutual attraction must be in equilibrium. Consequently, a mass of homogeneous fluid, the particles of which are mutually attractive, will always be in equilibrium when the free surface fulfils the single condition implied in the general equation obtained above. The same reasoning applies to the case of any fluid, elastic or incompressible.

If this demonstration be thought satisfactory, the question raised by Mr. Ivory, as to the sufficiency of the general equation, must be looked upon as settled. The suggestions here made with respect to the new condition tacitly introduced in Clairaut's reasoning, will, it is thought, enable us to trace the source of the difference of the view taken by Mr. Ivory, and that generally entertained. In one form or other, it seems to recur in every way in which that distinguished mathematician has treated the subject.

R. L. E.

VI.—ON THE CONDITION THAT A SURFACE MAY BE TOUCHED BY A PLANE IN A CURVE LINE.

In Vol. I. p. 83, a demonstration was given of a property of the Wave Surface, that it could be touched by the tangent plane in certain positions in a circle. This is a particular instance of what may be called a singular line in surfaces, analogous to a singular point in curved lines; and when the idea is generalized, it gives rise to the consideration of the possibility of surfaces being touched by a tangent plane in a continuous curve. It is proposed here to investigate the general analytical condition, that any points in a surface should possess this property.

Let $F(x, y, z) = 0$ be the equation to the surface, and put

$$\frac{dF}{dx} = L, \quad \frac{dF}{dy} = M, \quad \frac{dF}{dz} = N.$$

Then the equation to the tangent plane at any point x, y, z, is

$$Lx' + My' + Nz' = Lx + My + Nz.$$

Let the right-hand member of the equation be represented by P; then, if the plane touches the surface in a curve, $\frac{L}{P}, \frac{M}{P}, \frac{N}{P}$ remain constant while the coordinates vary subject to the condition $F(x, y, z) = 0$, and to another condition, which, together with that, determines the curve. It is this condition which we have to find.

Let
$$\frac{d^2F}{dx^2} = l, \qquad \frac{d^2F}{dy^2} = m, \qquad \frac{d^2F}{dz^2} = n,$$

$$\frac{d^2F}{dy\,dz} = \lambda, \qquad \frac{d^2F}{dx\,dz} = \mu, \qquad \frac{d^2F}{dx\,dy} = \nu.$$

Then, as $\frac{L}{P}, \frac{M}{P}, \frac{N}{P}$ are all constant,

$$\frac{dL}{L} = \frac{dM}{M} = \frac{dN}{N} = \frac{dP}{P} = dQ \text{ suppose.}$$

Or effecting the differentiation indicated,

$$\begin{aligned} dL &= l\,dx + \nu\,dy + \mu\,dz = L\,dQ, \\ dM &= \nu\,dx + m\,dy + \lambda\,dz = M\,dQ, \\ dN &= \mu\,dx + \lambda\,dy + n\,dz = N\,dQ. \end{aligned}$$

Eliminating dy and dz by cross multiplication,

$$R\,dx = \{L(mn - \lambda^2) + M(\lambda\mu - n\nu) - N(\lambda\nu - m\mu)\}\,dQ;$$

$$\text{where} \quad R = lmn - (l\lambda^2 + m\mu^2 + n\nu^2) + 2\lambda\mu\nu,$$

a symmetrical function of $l, m, n, \lambda, \mu, \nu$. Similarly we have

$$R\,dy = \{L(\lambda\mu - n\nu) + M(nl - \mu^2) + N(\mu\nu - l\lambda)\}\,dQ,$$
$$R\,dz = \{L(\lambda\nu - m\mu) + M(\mu\nu - l\lambda) + N(lm - \nu^2)\}\,dQ.$$

But from the equation to the surface we have also the condition

$$L\,dx + M\,dy + N\,dz = 0.$$

Therefore, multiplying the previous equations by L, M, N respectively, and adding, the first side of the equation disappears by the last condition, and we have

$$\begin{aligned} L^2(mn - \lambda^2) + M^2(ln - \mu^2) + N^2(lm - \nu^2) + 2MN(\mu\nu - l\lambda) \\ + 2LN(\lambda\nu - m\mu) + 2LM(\lambda\mu - n\nu) = 0; \end{aligned}$$

which equation, combined with $F(x, y, z) = 0$, determines the

curve of contact: and this condition must subsist in order that the surface may be touched by the tangent plane in a curve.

This expression may be reduced into a simpler shape, by supposing the original equation put into the form

$$f(x, y) - z = 0,$$

and employing the partial differential coefficients of z. We have then

$$L = p, \quad M = q, \quad N = -1, \quad l = r, \quad m = t, \quad n = 0,$$
$$\lambda = 0, \quad \mu = 0, \quad \nu = s;$$

and substituting these values in the equation, it is reduced to

$$rt - s^2 = 0.$$

This is the condition which subsists for every point of developable surfaces, as is easily seen ought to be the case, since in their case the tangent plane at every point touches them along a straight line.

S. S. G.

VII.—ON SINGULAR SOLUTIONS AND PARTICULAR INTEGRALS OF DIFFERENTIAL EQUATIONS.*

1. Any differential equation, of the n^{th} order and of the r^{th} degree in respect to the highest of its differential coefficients, may be conceived as resolved into r factors, each of the n^{th} order and of the first degree with regard to this differential coefficient, and the satisfaction of the compound equation will depend upon the satisfaction of the separate equations arising from putting each of these factors equal to zero. It will therefore be sufficient for us, in the following investigations, to discuss the nature of the singular solutions and the particular integrals of equations involving the highest differential coefficient of the first degree only.

Any differential equation, although it be multiplied by any function of the variables not involving the highest of the differential coefficients, is still regarded as the same differential equation.

2. Let $f\{x, y, y', \dots y^{(n)}\} = 0$, represent any differential equation of the n^{th} order and of the first degree in $y^{(n)}$. The following are evidently the only functions of $x, y, y', \dots y^{(n-1)}$ which can satisfy this identity.

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* From a Correspondent.

(α) A function, of which the differential coefficient is a factor of the differential equation, and of which the magnitude is any quantity, the variation of which is independent of the variation of the variables involved in the function. Let V represent such a function—then the identical equation $V = c$ is called the complete primitive, or a first integral of the differential equation, accordingly as the differential equation is of the first or of a higher order; and c, which represents the arbitrary magnitude of the function, is called the arbitrary constant. We shall always suppose that V contains no term independent of the variables, since any such term might be comprehended in the arbitrary constant. We will also use the term 'regular integral' to comprehend both complete primitives and first integrals. Every differential equation of the n^{th} order has (n) regular integrals.

(β) A function v of an assigned value (a), such that, after the differential equation has been so prepared that $v - a$ is not a factor of the whole of it, the differential coefficient of v is a factor of one portion of it, and $v - a$ of another.

The identity $v = a$ is called a particular integral, or a singular solution of the differential equation, accordingly as the relation which it establishes among the variables be or be not a result of the imposition of some definite value upon the arbitrary constant of the regular integral.

The term particular integral is applied also to identical equations belonging to functions of definite magnitude, which satisfy the differential equation precisely in the manner of the regular integral.

(γ) A function of invariable and definite magnitude which, by the relation which it establishes among the variables by the definiteness of its value, renders identically equal to zero a factor of the whole differential equation. If this function be not coincident with any of the functions of the second case, it is regarded as a factor foreign to the equation, and the relation which it establishes among the variables is not reckoned a solution.

3. Let $Z = 0$ represent a singular solution of a differential equation of the n^{th} order and of the first degree. The function Z will, as we know by the theory of equations, be equivalent to the product of a number of functions, in each of which the differential coefficient of highest order appears only in the first power, and the equating to zero of each of these factors will give us all the values of the highest differential coefficient in terms of the other variables. Hence, clearly, if v be any one of these factors, $v = 0$ will be a singular solution.

Let $c = \eta$ be a regular integral. Then, since the relation $v = 0$ is incompatible with a constant value for c, if we conjoin the relation $v = 0$ with the equation $c = \eta$, we shall get c equal to some function of $x, y, y', \ldots y^{(n-1)}$. Hence, evidently, if $v = 0$ be a singular solution, the most general expression for η may be re-

presented by $wv^{\alpha} + u$, where w and u are some functions of $x, y, y', \dots y^{(n-1)}$; and w has been so chosen that no term in u contains v either as a factor or as a divisor, and where the α, having been so chosen that w becomes neither zero nor infinity for the relation $v = 0$, is a positive quantity.

Differentiating the regular integral, we get

$$0 = \alpha v^{\alpha-1} w\, dv + v^{\alpha} dw + du,$$

$$\text{and } \therefore\ 0 = dv + \frac{v^{1-\alpha}}{\alpha}\left(v^{\alpha}\frac{dw}{w} + \frac{du}{w}\right);$$

but, since $v = 0$ is a singular solution of this equation, it is plain that α must be less than unity, since otherwise the equation would not be satisfied. Hence, if $c = wv^{\alpha} + u$ be a regular integral of a differential equation, the α being a positive quantity less than unity, $v = 0$ will be a singular solution of the differential equation.

Ex. If $c = (x + y + 1)^{\frac{1}{2}} + x$ be the complete integral of a differential equation, $x + y = -1$ or $x + y + 1 = 0$ will be a singular solution. Thus, differentiating, we have

$$0 = \frac{dx + dy}{2(x + y + 1)^{\frac{1}{2}}} + dx \dots\dots\dots\dots\dots (l),$$

$$\text{and } \therefore\ 0 = dx + dy + 2(x + y + 1)^{\frac{1}{2}} dx \dots (k),$$

which is satisfied by $x + y = -1$, a relation incompatible with any constant value for c in the integral.

4. The factor which renders the differential equation

$$0 = dv + \frac{v^{1-\alpha}}{\alpha}\left(v^{\alpha}\frac{dw}{w} + \frac{du}{w}\right)$$

a perfect differential, is $\alpha w v^{\alpha-1}$ or $\dfrac{\alpha w}{v^{1-\alpha}}$, which $= \infty$ when $v = 0$. Hence we see that the integrating factor of a differential equation becomes equal to infinity for the relation between $x, y, y', \dots$, expressed by a singular solution.

Thus, the integrating factor for equation (k) in the preceding article is $\dfrac{1}{2(x+y+1)^{\frac{1}{2}}}$, which $= \infty$ for the relation $x + y + 1 = 0$, which constitutes the singular solution.

5. It is clear that $v = 0$ does not satisfy the equation

$$0 = \alpha v^{\alpha-1} w\, dv + v^{\alpha} dw + du.$$

Hence we see, that every differential equation may be so prepared as to become insusceptible of any assigned singular solution; and likewise, that a state of perfect differentiality is an instance of this. Thus the equation (l) in Art. (3) is not satisfied by the singular solution $x + y + 1 = 0$.

6. We may write the differential equation in the form

$$dv + \frac{v}{a}\frac{dw}{w} + \frac{v^{1-\alpha}}{a}\frac{du}{w} = 0.$$

Here we see that $v = 0$ satisfies the equation when a is a negative quantity, as well as when it is a positive quantity less than (a); but $c = wv^{\alpha} + u$ becomes $= \infty$ in this case, which shews, that whenever such an equation is satisfied by $v = 0$, this must be a particular integral derivable from the regular integral, by putting the arbitrary constant equal to infinity.

Ex. Let $\qquad c = y(x^2 + y + a)^{-1} + by$

be the complete primitive.

The differential equation may be written

$$0 = (x^2 + y + a)\{dy + b\,dy\,(x^2 + y + a)\} - y(2x\,dx + dy),$$

and is satisfied by $x^2 + y + a = 0$, which gives $c = \infty$.

7. It may be that several singular solutions, such as $v = 0$, may correspond to the single value u of c.

Let $v_1 = 0$, $v_2 = 0$, ... $v_n = 0$, denote all the singular solutions of this class. Then clearly

$$c = \mu \,.\, \phi(u) \,.\, v_1^{\alpha_1} \,.\, v_2^{\alpha_2} \ldots v_n^{\alpha_n} + u,$$

where a_1, a_2, ... a_n are all positive quantities less than unity, where $\phi(u)$ is some function of u, and where μ is some function of the variables, such that $\mu = 0$ does not constitute a singular solution. Hence the most comprehensive expression for a singular solution for the value u of c in this case is

$$v_1^{\beta_1} \,.\, v^{\beta_2} \ldots v_n^{\beta_n} = 0,$$

where β_1, β_2, ... β_n, are any positive quantities.

Ex. For instance, let the complete integral of a differential equation be

$$c = (xy^2+1)^2 (x+y)^{\frac{1}{3}} (x^2+y^2-a^2)^{\frac{1}{2}} (x-y)^{\frac{1}{4}} (x^3-y)^{-1} + 3xy^2 + 1.$$

Then we know at once that the only singular solutions connected with the differential equation to which it belongs are

$$x + y = 0,$$
$$x^2 + y^2 - a^2 = 0,$$
$$x - y = 0;$$

or any product of these, for instance,

$$(x + y)^2 (x^2 + y^2 - a^2) = 0.$$

Again, $x^3 - y = 0$ is a particular integral corresponding to $c = \infty$. Also, $xy^2 + 1 = 0$ is a particular integral for $c = -2$.

8. Let $c = wv^{\alpha} + u$ be the regular integral of a differential equation, where the same character belongs to the symbols in-

volved as in Art. 3, except that here a is $=$ or > 1, and a positive quantity. Differentiating we get

$$0 = aw\, v^{x-1}\, dv + v^x\, dw + du.$$

Now, if we suppose $v = 0$, we get $du = 0$; but from the arrangements which were made about u this is impossible. Hence, when a is $=$ or > 1, $v = 0$ is no solution at all.

Ex. In equation $c = (x+y)^{\frac{3}{2}} \cdot (x-y)^{\frac{1}{2}} + x$,

$x + y = 0$ is no solution at all,

$x - y = 0$ is a singular solution.

9. Differentiating the equation $c = wv^x + u$ with respect to x alone, we have

$$\frac{dc}{dx} = aw\, v^{x-1}\, \frac{dv}{dx} + v^x\, \frac{dw}{dx} + \frac{du}{dx}.$$

Now, $\dfrac{dv}{dx}$ is not generally $= 0$ for all the simultaneous values of the variables of the equation $v = 0$. Hence, if $v = 0$ be either a singular solution or a particular integral, deducible from the regular integral by putting $c = \infty$, the imposition of the relation expressed by $v = 0$ upon the variables in the expression for $\dfrac{dc}{dx}$ will render it equal to infinity. In just the same way we may shew that $\dfrac{dc}{dy}$, $\dfrac{dc}{dy'}$, ..., for the relation $v = 0$, become all of them equal to infinity.

10. We will now proceed to shew, that every first integral of a differential equation gives rise to the same singular solutions.

Let $v = 0$ be a singular solution belonging to a first integral $c = wv^x + u$ of a differential equation

$$dv + \frac{v^{1-x}}{a}\left(v^x \frac{dw}{w} + \frac{du}{w}\right) = 0.$$

If this be not a singular solution connected with any other of the first integrals, it must be a particular integral, since it does satisfy the differential equation. Hence this other first integral must evidently be expressible under the form

$$c' = \mu v^\beta + b,$$

where b is some definite quantity, and where we will suppose β has been so chosen that μ does not equal 0 or ∞ for the relation $v = 0$. Differentiating this equation, we get

$$dv + \frac{v}{\beta}\,\frac{d\mu}{\mu} = 0\,;$$

and, since the differential equation must be the same for both first integrals, it follows that

$$\frac{v}{\beta}\frac{d\mu}{\mu} = \frac{v^{1-\alpha}}{\alpha}\left(v^{\alpha}\frac{dw}{w} + \frac{du}{w}\right);$$

therefore $\dfrac{v}{\beta}\dfrac{d\mu}{\mu} = \dfrac{v}{\alpha}\dfrac{dw}{w} + \dfrac{v^{1-\alpha}}{\alpha}\dfrac{du}{w}$;

therefore $v^{\alpha}\left(\dfrac{1}{\beta}\dfrac{d\mu}{\mu} - \dfrac{1}{\alpha}\dfrac{dw}{w}\right) = \dfrac{1}{\alpha}\dfrac{du}{w}$;

therefore $v^{\alpha} = \dfrac{\dfrac{1}{\alpha}\dfrac{du}{w}}{\dfrac{1}{\beta}\dfrac{d\mu}{\mu} - \dfrac{1}{\alpha}\dfrac{dw}{w}}$.

Hence, when $v = 0$, its equivalent $\dfrac{1}{\alpha}\dfrac{du}{w}\left(\dfrac{1}{\beta}\dfrac{d\mu}{\mu} - \dfrac{1}{\alpha}\dfrac{dw}{w}\right)^{-1}$ must also become equal to 0; but, from the arrangements which have been made with regard to the elements of this expression, it is clear that this condition cannot be satisfied for all the simultaneous values of the variables expressed by the equation $v = 0$. Hence we see that $v = 0$ must likewise be a singular solution with regard to every other first integral, or, in other words, that every first integral must give rise to the same singular solutions.

Ex. Let the complete integral of a differential equation of the second order be

$$c = (1 + a^2)x + ay^2 + x^2.$$

From which $0 = 1 + a^2 + 2ayy' + 2x$;

therefore $a^2 + 2ayy' + y^2y'^2 = y^2y'^2 - 2x - 1$;

whence $a = -yy' + (y^2y'^2 - 2x - 1)^{\frac{1}{2}}$;

therefore $y^2y'^2 - 2x - 1 = 0$ is a singular solution for one first integral, but

$$\begin{aligned} c &= (1 + a^2)x + ay^2 + x^2, \\ &= -x(2ayy' + 2x) + ay^2 + x^2, \\ &= a(y^2 - 2xyy') - x^2, \\ &= \{(y^2y'^2 - 2x - 1)^{\frac{1}{2}} - yy'\}(y^2 - 2xyy') - x^2; \end{aligned}$$

from which it is evident that $y^2y'^2 - 2x - 1 = 0$ is a singular solution for the other first integral.

11. Let $y = k$ be one of the values of y corresponding to a singular solution of a differential equation of the first order, k being some function of x.

Let the complete primitive be expressed under the form

$$c = w(y - k)^{\alpha} + u,$$

where α and w have been so chosen, that u contains no term involving $y - k$ as a factor, and that w is of the form

$$\rho_0 + \rho_1(y - k)^{\beta_1} + \rho_2(y - k)^{\beta_2} + \ldots$$

where $\rho_0, \rho_1, \rho_2, \ldots$ have not any of them $y-k$ as a constituent factor, and where, since $y=k$ belongs to a singular solution, $\beta_1, \beta_2, \ldots$ are all positive quantities.

Differentiating with respect to x, we get

$$0=\frac{dw}{dx}(y-k)^{\alpha}-\alpha w(y-k)^{\alpha-1}\frac{dk}{dx}+\frac{du}{dx}$$
$$+p\left\{\frac{dw}{dy}(y-k)^{\alpha}+\alpha w(y-k)^{\alpha-1}+\frac{du}{dy}\right\},$$

and therefore

$$-p=\frac{\frac{dw}{dx}(y-k)^{\alpha}-\alpha w(y-k)^{\alpha-1}\frac{dk}{dx}+\frac{du}{dx}}{\frac{dw}{dy}(y-k)^{\alpha}+\alpha w(y-k)^{\alpha-1}+\frac{du}{dy}},$$
$$=\frac{\frac{dw}{dx}(y-k)-\alpha w\frac{dk}{dx}+\frac{du}{dx}(y-k)^{1-\alpha}}{\frac{dw}{dy}(y-k)+\alpha w+\frac{du}{dy}(y-k)^{1-\alpha}},$$

therefore

$$-p+\frac{dk}{dx}=\frac{\left(\frac{dw}{dx}+\frac{dw}{dy}\frac{dk}{dx}\right)(y-k)+\left(\frac{du}{dx}+\frac{du}{dy}\frac{dk}{dx}\right)(y-k)^{1-\alpha}}{\alpha w+\frac{dw}{dy}(y-k)+\frac{du}{dy}(y-k)^{1-\alpha}},$$

but $\frac{dw}{dx}+\frac{dw}{dy}\frac{dk}{dx}=\frac{d\rho_0}{dx}+\frac{d\rho_0}{dy}\frac{dk}{dx}+\left(\frac{d\rho_1}{dx}+\frac{d\rho_1}{dy}\frac{dk}{dx}\right)(y-k)^{\beta_1}+\ldots$

Hence it is plain that we may write

$$p-\frac{dk}{dx}=\frac{\mathrm{P}(y-k)^{1-\alpha}+\mathrm{Q}(y-k)}{\mathrm{R}+\mathrm{S}(y-k)^{\gamma}},$$

where P, Q, R do not involve $y-k$ as a factor. Consequently, in the development of p for a substitution of $k+h$ for y, where h is an arbitrary quantity, and $y=k$ belongs to a singular solution, the index of the lowest power of h is fractional, and therefore $\frac{dp}{dy}=\infty$ for a singular solution

If $y-k=0$ had belonged to a constant value of c, not infinity, we should have got

$$-p+\frac{dk}{dx}=\frac{\left(\frac{dw}{dx}+\frac{dw}{dy}\frac{dk}{dx}\right)(y-k)}{\alpha w+\frac{dw}{dy}(y-k)},$$

and the expansion might have been effected in powers of h, of which the lowest index would have been not less than unity. Hence $\frac{dp}{dy}$ would not $=\infty$.

Again, if $y - k = 0$ had corresponded to an infinite value for c, we should have got

$$-p + \frac{dk}{dx} = \frac{\left(\frac{dw}{dx} + \frac{dw}{dy}\frac{dk}{dx}\right)(y-k) + \left(\frac{du}{dx} + \frac{du}{dy}\frac{dk}{dx}\right)(y-k)^{1+\alpha}}{aw + \frac{dw}{dy}(y-k) + \frac{du}{dy}(y-k)^{1+\alpha}},$$

and in the development by powers of h the lowest index of h would have been unity. Hence we see that $\frac{dp}{dy}$ would not $= \infty$.

From these results we see, that whenever we discover a solution of a differential equation, we may ascertain whether or not it is a singular solution by trying whether it will render $\frac{dp}{dy} = \infty$.

It must be remarked, that if we determine $\frac{dp}{dy}$ from a differential equation, and make the result $= \infty$, it does not follow that the relation between x and y, resulting from this condition, will be a singular solution—it may not be a solution at all; but if it be a solution, as we can ascertain by seeing whether it satisfies the equation, it must be a singular solution. So that if there be any singular solution or singular solutions, we are certain to detect them by this method.

Ex. 1. Let us take the differential equation

$$0 = (x^2 - y)\, dy + 3x\, dx,$$

$$\text{we get } \frac{dy}{dx} = \frac{3x}{y - x^2};$$

$$\text{therefore } \frac{d}{dy}\frac{dy}{dx} = -\frac{3x}{(y - x^2)^2} = \infty \text{ when } y = x^2;$$

but $y = x^2$ does not satisfy the differential equation, and is no solution at all.

Ex. 2. Let us take the differential equation

$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} + x = 0,$$

$y^2 + (x - 1)^2 = 0$ satisfies this equation.

Solving it in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = -\tfrac{1}{2}y + \tfrac{1}{2}\sqrt{y^2 - 4x};$$

$$\text{therefore } \frac{d}{dy}\frac{dy}{dx} = -\tfrac{1}{2} + \tfrac{1}{2}\frac{y}{\sqrt{y^2 - 4x}},$$

which does not $= \infty$, for $y^2 + (x - 1)^2 = 0$.

Hence, $y^2 + (x - 1)^2 = 0$ is a particular integral.

Again, equating $y^2 - 4x$ to zero, we get $y = \pm 2\sqrt{x}$; and substituting this in our differential equation, we have

$$x + 2x + x = 0, \text{ an absurdity.}$$

Hence the differential equation has no singular solutions.

Ex. 3. If from the equation

$$x\frac{dy}{dx} - y = y\frac{dy}{dx} + (a - x)\frac{dy^2}{dx^2},$$

we determine $\frac{d}{dy}\frac{dy}{dx}$, and equate it to infinity, we shall get

$$(x + y)^2 - 4ay = 0,$$

and this satisfies the equation. Hence it is a singular solution.

12. Let

$$c = \phi(u) \cdot \mu \cdot v_1^{\alpha_1} \cdot v_2^{\alpha_2} \ldots\ldots v_n^{\alpha_n} + u$$

be a regular integral of a differential equation, the symbols here involved being such as in Art. 7.

From this equation suppose that we obtained another shape of the regular integral $V = f(x, y, y', \ldots y^{(n-1)}, c) = 0$; and let us suppose this function to be such, that $V_{c=u} = 0$ comprehends no relations among the variables, except those of the singular solutions $v_1 = 0$, $v_2 = 0$, &c. Then clearly V must be such a function of the variables, that when u is substituted in it for c, it shall be reduced to $v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n}$, where $\beta_1, \beta_2, \ldots \beta_n$ are all positive quantities.

Hence, clearly,

$$V = (v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n})\lambda - \left\{\frac{(c - u)\mu}{\phi(u)} . v_1^{\beta_1 - \alpha_1} . v_2^{\beta_2 - \alpha_2} \ldots v_n^{\beta_n - \alpha_n}\right\}^\lambda,$$

where λ is some positive quantity.

From this we get

$$\frac{dV}{dc} = -\lambda(c - u)^{\lambda - 1} . \left(\frac{\mu}{\phi(u)} . v_1^{\beta_1 - \alpha_1} \ldots v_n^{\beta_n - \alpha_n}\right)^\lambda,$$

$$= 0 \text{ when } c = u, \text{ provided that } \lambda > 1.$$

Hence we see, that if $V = 0$ be a regular integral of a differential equation, involving the arbitrary constant c to a higher power than the first, we may sometimes get a singular solution by obtaining a value of c from the equation $\frac{dV}{dc} = 0$, and substituting it in the equation $V = 0$.

If V had been

$$= [\mu^\gamma \{\phi(u)\}^\delta . v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n}]^\lambda -$$
$$- [(c - u)\mu^{1+\gamma} . \{\phi(u)\}^{1+\delta} . v_1^{\beta_1 - \alpha_1} \ldots v_n^{\beta_n - \alpha_n}],$$

then clearly $\frac{dV}{dc} = 0$ would have given us, not only singular solutions, but also solutions $\mu = 0$, $\phi(u) = 0$, of which the former are no solutions at all, and the latter particular integrals.

Ex. Let $c = (x + y)(x - y)^{\frac{1}{2}} + x$,

$$(c - x)^2 = (x + y)^2 (x - y),$$

$$V = (c - x)^2 - (x + y)^2 (x - y).$$

Put
$$\frac{dV}{dc} = 2(c - x) = 0;$$

therefore $c = x$,

and we have
$$(x + y)^2 (x - y) = 0.$$

Now the differential equation is

$$0 = (dx + dy)(x - y)^{\frac{1}{2}} + (x + y)\frac{dx - dy}{2(x - y)^{\frac{1}{2}}} + dx,$$

and $x + y = 0$ does not satisfy the equation.

Hence $(x + y)^2 (x - y) = 0$ gives not only a relation $x - y = 0$, constituting a singular solution, but also $x + y = 0$, which is no solution at all. Thus we see, that unless we can obtain the differential equation, or, which is the same thing, determine another form of $V = 0$, where the arbitrary constant is explicit, we do not know whether $\frac{dV}{dc} = 0$ will lead to any kind of solution or not. And if we *can* get c explicit, we can always see at once what the singular solutions are.

13. Let $f(x, y, c) = 0$ be the complete primitive of a differential equation of the first order, the arbitrary constant being involved among the variables.

Differentiating, we get

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = 0;$$

but since c is a function of the two independent quantities x and y, we have

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dc}\frac{dc}{dx} = 0,$$

and
$$\frac{df(x, y, c)}{dy} + \frac{df(x, y, c)}{dc}\frac{dc}{dy} = 0;$$

and from these three equations we get

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = -\frac{dV}{dc}\left(\frac{dc}{dx} + \frac{dc}{dy}\frac{dy}{dx}\right) \ldots(k).$$

Suppose next that c has some variable value u, incompatible

with the expression for c, resulting from the solution of $f(x, y, c)=0$ in respect to c. Differentiating equation $f(x, y, u) = 0$, we have

$$\frac{df(x, y, u)}{dx} + \frac{df(x, y, u)}{dy}\frac{dy}{dx} + \frac{df(x, y, u)}{du}\left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right) = 0.$$

Suppose, now, that $\frac{df(x, y, u)}{du} = 0$; then, since $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}$ cannot be generally equal to an infinite quantity, since u does not involve as a factor the function, the vanishing of which reduced c to u, we shall have

$$\frac{df(x, y, u)}{dx} + \frac{df(x, y, u)}{dy}\frac{dy}{dx} = 0;$$

but u is involved in $f(x, y, u) = 0$, just in the same way as c was in $f(x, y, c) = 0$; and therefore from these two equations we shall get the same expressions for u and for c. Consequently the equation resulting from the differentiation of $f(x, y, c) = 0$, and the subsequent elimination of c is satisfied by the equation $f(x, y, u) = 0$, provided that $\frac{df(x, y, u)}{du} = 0$. We must not, however, be led to infer from this, that $f(x, y, u) = 0$ is necessarily a solution of the differential equation belonging to the complete primitive $f(x, y, c) = 0$; for by equation (k) we see, that the differential equation which is satisfied by $f(x, y, u) = 0$ is the differential equation in its state of perfect differentiality multiplied by $\frac{df(x, y, u)}{du}$, and therefore its satisfaction may result from the fact, that $\frac{df(x, y, u)}{du}$ has been made $= 0$ quite independently of its own peculiar constitution.

The same remarks are applicable to a first integral

$$f(x, y, y', \dots y^{(n-1)} c) = 0$$

of any equation.

Ex. Let $c = (x + y)^2 + y$ be the complete primitive of a differential equation.

$$(c - y)^2 - (x + y)^4 = 0 = f(x, y, c) \text{ suppose},$$

$$\frac{df(x, y, c)}{dc} = 2(c - y) = 0 \text{ suppose};$$

$$\text{therefore } c = y,$$

$$\text{therefore } f(x, y, u) = -(x + y)^4;$$

$$\text{but } \frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = 0,$$

$$\text{therefore} \quad -4(x + y)^3 - \{2(c - y) + 4(x + y)\}\frac{dy}{dx} = 0,$$

$$\text{and} \quad -4(x+y)^3 - \{2(x+y)^2 + 4(x+y)\}\frac{dy}{dx} = 0,$$

and this is *satisfied* by $x + y = 0$,

but $x + y$ is a factor. Dividing out by it, we get

$$-4(x+y)^2 - \{2(x+y) + 4\}\frac{dy}{dx} = 0,$$

which is not satisfied by $x + y = 0$, and therefore $(x + y)^4 = 0$ or $x + y = 0$ is not a solution.

W. W.

VIII.—ON SOME EXPRESSIONS FOR THE AREA OF A TRIANGLE.

The expressions for the area of a triangle are usually given in terms of the sides and angles which, being the fundamental parts of the figure, are naturally the quantities of which every expression relating to the triangle is made a function. It is, however, possible to express the area of a triangle in terms of other independents (if we may use the phrase); and as in two cases the results are remarkable for simplicity, we shall here proceed to investigate them. The independent quantities which we shall assume in the first case, are the lines joining the angles with the middle points of the opposite sides, and it will be seen that the form of the expression is exactly the same as that involving the sides.

1. Let ABC (fig. 3.) be a triangle, D, E, F the middle points of the sides; join AD, BE, CF, which will all pass through one point O, such that $OD = \frac{1}{2}OA$, $OE = \frac{1}{2}OB$, $OF = \frac{1}{2}OC$. Produce BO to G, and make $OG = BO$, so that $OE = EG$, and join AG, CG. Now, since $AE = CE$ and $OE = EG$, the triangles AEO, CEG are equal in every respect, and hence CG is parallel and equal to AO; similarly, AG is parallel and equal to OC. Now, the three triangles BOC, AOC, AOB, being all equal, each is equal to $\frac{1}{3}$ABC; but AOG = AOC, as they are on the same base and between the same parallels; therefore AOG = $\frac{1}{3}$ABC.

Let $AO = \alpha$, $BO = OG = \beta$, $OC = AG = \gamma$. Then

$$\text{area AOG} = \sqrt{\frac{(\alpha+\beta+\gamma)}{2}\frac{(\alpha+\beta-\gamma)}{2}\frac{(\alpha+\gamma-\beta)}{2}\frac{(\beta+\gamma-\alpha)}{2}},$$

Let $AD = h$, $BE = k$, $CF = l$. Then

$$\alpha = \tfrac{2}{3}h, \quad \beta = \tfrac{2}{3}k, \quad \gamma = \tfrac{2}{3}l:$$

substituting these values in the previous expression,

$$\text{area AOG} = \sqrt{\frac{(h+k+l)}{3}\frac{(h+k-l)}{3}\frac{(h+l-k)}{3}\frac{(l+k-h)}{3}};$$

and as area ABC = 3 area AOG,

$$\text{area ABC} = \tfrac{1}{3}\sqrt{(h+k+l)(h+k-l)(h+l-k)(k+l-h)}.$$

If we put $h+k+l=2s$, and transform accordingly the other factors, we find

$$\text{area ABC} = \tfrac{4}{3}\sqrt{s(s-h)(s-k)(s-l)},$$

which is of the same form as the expression for the area in terms of the sides.

2. We cannot obtain a similar expression in terms of the perpendiculars from the angles on the opposite sides; but if we avail ourselves of the relations between these and the radii of the circles touching the sides, given in vol. I. p. 21, we obtain a very simple expression for the area of the triangle.

Let p, q, r be the three perpendiculars on the sides a, b, c respectively; then, if A be the area of the triangle,

$$a = \frac{2A}{p}, \quad b = \frac{2A}{q}, \quad c = \frac{2A}{r}.$$

Now, $$A = \sqrt{\frac{(a+b+c)}{2}\,\frac{(a+b-c)}{2}\,\frac{(a+c-b)}{2}\,\frac{(b+c-a)}{2}};$$

and substituting for a, b, c their values,

$$A = A^2\sqrt{\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{q}\right)\left(\frac{1}{q}+\frac{1}{r}-\frac{1}{p}\right)}.$$

But if ρ be the radius of the inscribed circle, ρ_1, ρ_2, ρ_3 the radii of the circles which touch two of the sides internally and one externally, it was shown, in the article referred to above, that

$$\frac{1}{\rho} = \frac{1}{p}+\frac{1}{q}+\frac{1}{r}, \quad \frac{1}{\rho_1} = \frac{1}{p}+\frac{1}{q}-\frac{1}{r},$$
$$\frac{1}{\rho_2} = \frac{1}{p}+\frac{1}{r}-\frac{1}{q}, \quad \frac{1}{\rho_3} = \frac{1}{q}+\frac{1}{r}-\frac{1}{p};$$

$$\text{therefore } A = \frac{A^2}{\sqrt{\rho\rho_1\rho_2\rho_3}},$$

$$\text{or } A = \sqrt{\rho\rho_1\rho_2\rho_3}.$$

From this we can obtain easily an expression for the perimeter of the triangle in terms of these radii. For since we have $\rho(a+b+c) = 2A$, it follows that

$$a+b+c = 2\sqrt{\frac{\rho_1\rho_2\rho_3}{\rho}}.$$

W.

IX.—ON THE METHOD OF SPHERICAL COORDINATES. No. II.

We shall now proceed, in continuation of Art. I. of No. V., to further exemplifications of the method of spherical coordinates, applying it first to find the areas and lengths of spherical curves.

16. Let $C\phi$, $C\psi$ (fig. 4.) be the coordinate axes, and let two consecutive ordinates and parallels to $C\phi$ form a small rectangular area, which may ultimately be considered as a plane area bounded by rectilinear sides. Then

The sides of this parallel to $C\phi$ will be $\cos\psi\, d\phi$,
.... $C\psi$ .... $d\psi$,

and therefore the element of the area will be $\cos\psi\, d\psi\, d\phi$; so that if A be the area,

$$A = \iint \cos\psi\, d\psi\, d\phi = \int \sin\psi\, d\phi \text{ .... (1).}$$

If, therefore, we have, by means of the equation to the curve, a relation between ψ and ϕ, by substituting for ψ its value in terms of ϕ, or conversely, and integrating between the proper limits, we can determine the area.

As an example, let us take the curve whose equation is

$$\psi = \phi.$$

$$\text{Then } A = \int \sin\phi\, d\phi = C - \cos\phi;$$

which, taken from $\phi = 0$ to $\phi = \frac{\pi}{2}$, gives $A = 1$, A being here the fourth part of the area included between the curve and the axis of ϕ. The area included between the curve and the axis of ψ is $\frac{\pi}{2} - 1$, and therefore the whole area enclosed by the curve being four times this, is $2\pi - 4$; which, subtracted from the hemisphere, gives for the residue the square of the diameter of the sphere. Now, the curve whose equation is $\phi = \psi$, is by (12) that produced by the intersection with the sphere of a cylinder whose radius is half the radius of the sphere, and whose circumference passes through the centre of the sphere; therefore this cylinder cuts off from the sphere such an area, that the residue of the hemisphere is quadrable. This celebrated proposition was proposed by Viviani as a challenge to the mathematicians of his day, and was solved by James Bernouilli.

17. If we refer the curves to polar coordinates ϕ and θ, we have for the sides of the elemental area

$$d\phi \text{ and } \sin\phi\, d\theta;$$

and therefore if A be the area,

$$\begin{aligned} A &= \iint \sin\phi\, d\theta\, d\phi, \\ &= \int (C - \cos\phi)\, d\theta. \end{aligned}$$

Now, if when $\phi = 0$, $A = 0$,

$$0 = \int (C - 1)\, d\theta,$$

whence $C = 1$; therefore

$$A = \int (1 - \cos\phi)\, d\theta.$$

As an example of the application of this formula, let us take the Loxodrome, the equation to which is

$$\tan \tfrac{1}{2}\phi = \epsilon^{\theta \cot \alpha},$$

$$\text{or } \log \tan \tfrac{1}{2}\phi = \theta \cot \alpha;$$

$$\text{whence } \frac{d\phi}{\sin\phi} = d\theta \cot \alpha;$$

$$\text{therefore } A = \tan \alpha \int \left(\frac{1 - \cos\phi}{\sin\phi}\right) d\phi,$$

$$= 2 \tan \alpha \int \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}}\, d\frac{\phi}{2};$$

$$\text{therefore } A = C - 2 \tan \alpha \log \left(\cos \frac{\phi}{2}\right),$$

$$\text{when } \phi = 0, \quad \cos\frac{\phi}{2} = 1, \quad \log\left(\cos\frac{\phi}{2}\right) = 0,$$

$$\text{when } \phi = \frac{\pi}{2}, \quad \cos\frac{\phi}{2} = \frac{1}{\sqrt{2}}.$$

Therefore from $\phi = 0$ to $\phi = \dfrac{\pi}{2}$,

$$A = -2 \tan \alpha \log \frac{1}{\sqrt{2}},$$

$$= 2 \tan \alpha \log \sqrt{2} = \tan \alpha \log 2.$$

18. To find the differential expression for the length of a spherical curve.

Referring the curve to rectangular axes (fig. 5.), let $CM = \phi$, $PM = \psi$, and let PP', an element of the curve, $= d\sigma$. Then, as

$$PP'^2 = Pp^2 + P'p^2 \text{ ultimately},$$

$$d\sigma^2 = (\cos\psi)^2\, d\phi^2 + d\psi^2;$$

$$\text{therefore } \frac{d\sigma}{d\phi} = \sqrt{(\cos\psi)^2 + \left(\frac{d\psi}{d\phi}\right)^2}.$$

If we suppose the curve referred to polar coordinates, it is easily seen that

$$\frac{d\sigma}{d\theta} = \sqrt{(\sin\phi)^2 + \left(\frac{d\phi}{d\theta}\right)^2}.$$

If the equation to the curve be

$$\phi = m\theta, \ \frac{d\phi}{d\theta} = m, \ \text{and} \ \frac{d\sigma}{d\theta} = \sqrt{m^2 + (\sin m\theta)^2},$$

which can only be integrated by elliptic functions.

If the curve be the Loxodrome, the equation to which is

$$\tan\frac{\phi}{2} = e^{\theta \cot \alpha},$$

$$\frac{d\phi}{\sin\phi} = d\theta \cot \alpha,$$

$$\text{therefore } d\theta^2 = d\phi^2 \frac{(\tan \alpha)^2}{(\sin \phi)^2}.$$

$$\text{Whence, } d\sigma = d\phi \sec \alpha,$$

$$\text{and } \sigma = \phi \sec \alpha + C;$$

which, if taken from $\phi = 0$ to $\phi = \pi$, corresponding to $\theta = -\infty$ and $\theta = \infty$, gives

$$\sigma = \pi \sec \alpha.$$

19. To find the volume of the solid contained between the surface of the sphere and the cylindrical surface passing through any curve of the sphere, and perpendicular to the plane of the equator.

Let PM (fig. 7.) be an elemental prism of the solid, ECM $= \phi$, PCM $= \psi$, $r =$ radius of the sphere. Then

$$\text{height of prism} = r \sin \psi,$$

$$\text{base of prism} = -r \cos \psi \, d\phi \, . \, d(r \cos \psi),$$

(the negative sign being taken as the solid is measured from the surface of the sphere).

$$\begin{aligned} \text{Therefore content of prism} &= -r^2 \sin \psi \cos \psi \, d\phi \, d(r \cos \psi), \\ &= r^3 \sin^2 \psi \cos \psi \, d\psi \, d\phi; \\ \text{therefore solid} &= r^3 \iint \sin^2 \psi \cos \psi \, d\psi \, d\phi, \\ &= \tfrac{1}{3} r^3 \int (\sin \psi)^3 \, d\phi. \end{aligned}$$

As an example, take the curve formed by the intersection of a circular cylinder passing through the centre of the sphere, its equation being $\phi = \psi$,

$$\begin{aligned} \text{solid} &= \tfrac{1}{3} r^3 \int (\sin \phi)^3 \, d\phi, \\ &= \tfrac{1}{3} r^3 \int d\phi \, \{1 - (\cos\phi)^2\} \sin \phi, \\ &= \tfrac{1}{3} r^3 \{\cos\phi - \tfrac{1}{3}(\cos\phi)^3\} \, C; \end{aligned}$$

and taking this from $\phi = 0$ to $\phi = \dfrac{\pi}{2}$.

$$\text{solid} \doteq \frac{2}{9} r^3.$$

20. We shall now consider problems connected with the tangencies of spherical curves; and in the first place, as in plane

curves we find the condition for a straight line being a tangent, so we shall investigate the equation to a great circle which touches a spherical curve.

Let the coordinates of the point of contact be ϕ_1, ψ_1: then by (1) the equation to a great circle passing through it is

$$\tan\psi\sin(\phi_1 - a) = \tan\psi_1\sin(\phi - a).$$

Now, as the circle is to touch the curve, this equation must hold for the point $\phi_1 + d\phi_1$, $\psi_1 + d\psi_1$. Hence, taking the logarithmic differential, we have

$$\frac{\cos(\phi_1 - a)}{\sin(\phi_1 - a)}d\phi_1 = \frac{(\sec\psi_1)^2}{\tan\psi_1}d\psi_1,$$

$$\text{and therefore } \cot(\phi_1 - a) = \frac{(\sec\psi_1)^2}{\tan\psi_1}\frac{d\psi_1}{d\phi_1}.$$

But as $\sin(\phi - a) = \sin(\phi - \phi_1 + \phi_1 - a)$

$$= \sin(\phi - \phi_1)\cos(\phi_1 - a) + \cos(\phi - \phi_1)\sin(\phi_1 - a),$$

$$\tan\psi = \tan\psi_1\{\sin(\phi - \phi_1)\cot(\phi_1 - a) + \cos(\phi - \phi_1)\}.$$

Whence, eliminating $\cot(\phi_1 - a)$,

$$\tan\psi = (\sec\psi_1)^2\frac{d\psi_1}{d\phi_1}\sin(\phi - \phi_1) + \tan\psi_1\cos(\phi - \phi_1),$$

which is the required equation.

21. The equation to the normal circle is readily found from the condition, that it shall be perpendicular to the tangent circle. Now, by (5) the equation to a circle passing through a point (ϕ_1, ψ_1), and perpendicular to a circle whose equation is

$$\tan\psi = m\sin(\phi - a),$$

is $\tan\psi = -\dfrac{1}{m\cos(\phi_1 - a)}\{\sin(\phi - \phi_1) - m\tan\psi_1\cos(\phi - a)\}.$

In this case,

$$m = \tan \text{MTN} \text{ and } a - \phi_1 = \text{NT}, \text{ (fig. 6)}.$$

Now, by Napier's rules in the triangle MNT,

$$\sin \text{NT} = \tan\psi_1\cot \text{MTN};$$

$$\text{therefore } \frac{1}{m} = -\frac{\sin(\phi_1 - a)}{\tan\psi_1},$$

$$\text{and } \frac{1}{m\cos(\phi_1 - a)} = -\frac{\tan(\phi_1 - a)}{\tan\psi_1} = (\cos\psi_1)^2\frac{d\phi_1}{d\psi_1}.$$

Also, as $\cos(\phi - a) = \cos(\phi - \phi_1 + \phi_1 - a)$,

$$\frac{\cos(\phi - a)}{\cos(\phi_1 - a)} = \cos(\phi - \phi_1) - \tan(\phi_1 - a)\sin(\phi - \phi_1)$$

$$= \cos(\phi - \phi_1) - \sin\psi_1\cos\psi_1\sin(\phi - \phi_1)\frac{d\phi_1}{d\psi_1}.$$

Substituting these values, we find

$$\tan \psi = -\frac{d\phi_1}{d\psi_1} \sin(\phi - \phi_1) + \tan \psi_1 \cos(\phi - \phi_1),$$

which is the required equation.

22. From these equations, with the aid of Napier's rules, we can easily determine the values of the various parts connected with the tangent and normal.

Let QMQ′ (fig. 6.) be the curve, MT the tangent circle, MK the normal circle, at the point M (ϕ_1, ψ_1). Then

$$\text{CM} = \phi_1, \quad \text{MN} = \psi_1.$$

If in the equation to the tangent we make $\psi = 0$, we have

$$\tan(\phi - \phi_1) = \tan \text{NT} = -\sin \psi_1 \cos \psi_1 \frac{d\phi_1}{d\psi_1},$$

which determines NT, and therefore CT.

If we make $\phi = 0$, we have

$$\tan \psi = \tan \text{CT}' = \tan \psi_1 \cos \phi_1 - (\sec \psi_1)^2 \sin \phi_1 \frac{d\psi_1}{d\phi_1}.$$

In the triangle MNT we have, by Napier's rules,

$$\sin \text{MN} = \tan \text{NT} \cot \text{NMT};$$

$$\text{therefore } \tan \text{NMT} = -\cos \psi_1 \frac{d\phi_1}{d\psi_1}.$$

In the equation to the normal, if we make $\psi = 0$, we have

$$\tan(\phi - \phi_1) = -\tan \text{NK} = \tan \psi_1 \frac{d\psi_1}{d\phi_1}.$$

23. If we refer the curve to polar coordinates, P being the pole, and if we put $\text{PM} = \phi_1$, $\text{CPM} = \theta_1$, we shall easily find the following values of the different parts of the figure, PLM being drawn perpendicular to the tangent and XPS to the radius vector.

$$\tan \text{LPM} = \frac{1}{\sin \phi_1 \cos \phi_1} \frac{d\phi_1}{d\theta_1},$$

$$\tan \text{PS} = \sin^2 \phi_1 \frac{d\theta_1}{d\phi_1}, \quad \sin \text{PL} = \frac{\sin^2 \phi_1}{\sqrt{\sin^2 \phi_1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}},$$

$$\tan \text{ML} = \frac{\tan \phi_1}{\sqrt{\sin^2 \phi_1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}},$$

$$\cos \text{MX} = \frac{\cos \phi_1}{\sqrt{1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}}, \quad \tan \text{PX} = \frac{d\phi_1}{d\theta_1},$$

24. Having discussed the principal points of the theory of tangents, we shall now consider the curvature of spherical curves. This, as in plane curves, will be determined by the radius of the small circle of the sphere which has a contact of the second order with the given curve, so that the principle of the investigation is the same as that in plane curves, though the forms of the equations make the expressions much more complicated. The equation to a small circle whose radius is γ, and the coordinates of the pole of which are a, β, is by (4),

$$\cos\gamma = \cos\beta\cos\psi\cos(\phi - a) + \sin\beta\sin\psi \ldots\ldots (1).$$

Differentiating this, considering ϕ and ψ as variable, we have

$$0 = \cos\beta\,\{\sin\psi\cos(\phi - a)\,d\psi + \cos\psi\sin(\phi - a)\,d\phi\} \\ - \sin\beta\cos\psi\,d\psi\,;$$

$$\text{whence}\quad \tan\beta = \tan\psi\cos(\phi - a) + \sin(\phi - a)\frac{d\phi}{d\psi} \ldots\ldots (2).$$

Differentiating a second time,

$$0 = \sec^2\psi\cos(\phi - a) - \tan\psi\sin(\phi - a)\frac{d\phi}{d\psi} \\ + \cos(\phi - a)\left(\frac{d\phi}{d\psi}\right)^2 + \sin(\phi - a)\frac{d^2\phi}{d\psi^2},$$

which may be put under the form

$$A\cos(\phi - a) - B\sin(\phi - a) = 0 \ldots\ldots (3),$$

$$\text{where}\quad A = (\sec\psi)^2 + \left(\frac{d\phi}{d\psi}\right)^2,$$

$$\text{and}\quad B = \tan\psi\,\frac{d\phi}{d\psi} - \frac{d^2\phi}{d\psi^2}.$$

Now, from (3) we have

$$\tan(\phi - a) = \frac{A}{B},$$

$$\text{therefore}\quad \cos(\phi - a) = \frac{B}{\sqrt{A^2 + B^2}},$$

$$\text{and}\quad \sin(\phi - a) = \frac{A}{\sqrt{A^2 + B^2}}.$$

Substituting these in (2), we find

$$\tan\beta = \frac{B\tan\psi + A\,\dfrac{d\phi}{d\psi}}{\sqrt{A^2 + B^2}} \ldots\ldots\ldots (4).$$

Also, equation (3) may be put under the form

$$A(\cos\phi\cos a + \sin\phi\sin a) \\ - B(\sin\phi\cos a - \cos\phi\sin a) = 0\,;$$

whence

$$\tan a = -\frac{A\cos\phi - B\sin\phi}{A\sin\phi + B\cos\phi} \;\ldots\ldots\; (5);$$

and thus the coordinates of the centre of curvature are determined. To find the value of γ, we have from (4)

$$\sin\beta = \frac{B\tan\psi + A\dfrac{d\phi}{d\psi}}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}$$

$$\text{and } \cos\beta = \frac{\sqrt{A^2+B^2}}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}.$$

Substituting these values, and that of $\cos(\phi - a)$ in (1), we have

$$\cos\gamma = \frac{B\cos\psi + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)\sin\psi}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}$$

$$= \frac{B + A\sin\psi\cos\psi\dfrac{d\phi}{d\psi}}{\sqrt{A^2\cos^2\psi\left\{1 + \left(\dfrac{d\phi}{d\psi}\right)^2\right\} + 2AB\sin\psi\cos\psi + B^2}} \;\ldots(6).$$

25. As an example of the application of these formulæ, let us take the curve, the equation to which is

$$\phi = \psi;$$

$$\text{whence } \frac{d\phi}{d\psi} = 1, \quad \frac{d^2\phi}{d\psi^2} = 0.$$

This gives

$$A = 1 + (\sec\psi)^2, \qquad B = \tan\psi;$$

and substituting these in (4), we find

$$\tan\beta = \frac{2}{\sqrt{1 + 3(\cos\phi)^2}}, \quad \text{since } \phi = \psi.$$

Also, substituting these values of A and B in (5), and observing that from the equation to the curve $\phi = \psi$, we obtain, after some reductions,

$$\tan a = -\frac{2\cot\phi}{2 + (\sec\phi)^2},$$

$$\text{or } \cot a = -\tfrac{1}{2}\tan\phi\,\{3 + (\tan\phi)^2\}.$$

Lastly, substituting the same values in (6) after reducing, we obtain

$$\cos\gamma = \frac{\sin\phi\,\{2 + (\cos\phi)^2\}}{\sqrt{5 + 3(\cos\phi)^2}};$$

$$\text{when } \phi = 0, \quad \gamma = \frac{\pi}{2}; \quad \text{when } \phi = \frac{\pi}{2}, \quad \cos\gamma = \frac{2}{\sqrt{5}}.$$

26. The equation to the evolute of a spherical curve will be found in the same way as that of a plane curve, by eliminating ϕ and ψ between (4) and (5). If we take as an example the same curve as before,

$$\phi = \psi,$$

we have
$$4(\cot\beta)^2 = 1 + 3(\cos\phi)^2,$$
$$\text{and } 4(\cot\alpha)^2 = \tan^2\phi\,\{3 + (\tan\phi)^2\}^2;$$
$$\text{therefore } (\cos\phi)^2 = \frac{4(\cot\beta)^2 - 1}{3},$$
$$\text{and } (\tan\phi)^2 = \frac{4\{1 - (\cot\beta)^2\}}{4(\cot\beta)^2 - 1}.$$

Substituting this value,

$$(\cot\alpha)^2 = \frac{\{1 - (\cot\beta)^2\}\{8(\cot\beta)^2 - 1\}^2}{\{4(\cot\beta)^2 - 1\}^3},$$
$$\text{or } (\tan\alpha)^2 = \frac{\{4 - (\tan\beta)^2\}^3}{\{(\tan\beta)^2 - 1\}\{8 - (\tan\beta)^2\}^2},$$

which is the equation to the evolute.

27. As another example, let us take the Loxodrome. This curve is best referred to polar coordinates, and we must therefore adapt our formulæ to the change, which is easily done by putting for ψ and β, $\frac{\pi}{2} - \psi$ and $\frac{\pi}{2} - \beta$, and leaving ϕ and α unchanged. The equation to the Loxodrome is

$$\tan\frac{\psi}{2} = \epsilon^{\phi\cot c};$$

$$\text{whence } \frac{d\psi}{\sin\psi} = d\phi\cot c;$$

which gives us

$$A = \frac{\{1 + (\tan c)^2\}}{(\sin\psi)^2} = \frac{(\sec c)^2}{(\sin\psi)^2}, \quad B = 0.$$

$$\text{whence } \cot\beta = \frac{\tan c}{\sin\psi},$$

$$\tan\alpha = -\cot\phi, \text{ or } \alpha = \frac{\pi}{2} + \phi,$$

$$\text{and } \cos\gamma = \frac{\tan c \sin\psi}{\sqrt{(\sin\psi)^2 + (\tan c)^2}},$$

which determines the radius of curvature.

For the evolute we have

$$\tan\beta \tan c = \sin\psi = \frac{2\tan\frac{\psi}{2}}{1 + \left(\tan\frac{\psi}{2}\right)^2}.$$

Putting for $\tan\frac{\psi}{2}$ its value from the equation to the curve, and putting $\alpha - \frac{\pi}{2}$ for ϕ, we get

$$\tan\beta \tan c = \frac{2\epsilon^{\left(\alpha - \frac{\pi}{2}\right)\cot c}}{1 + \epsilon^{2\left(\alpha - \frac{\pi}{2}\right)\cot c}};$$

or dividing the numerator and denominator by $\epsilon^{\left(\alpha - \frac{\pi}{2}\right)\cot c}$, and inverting,

$$\cot\beta \cot c = \tfrac{1}{2}\left\{\epsilon^{\left(\alpha - \frac{\pi}{2}\right)\cot c} + \epsilon^{-\left(\alpha - \frac{\pi}{2}\right)\cot c}\right\},$$

which is the equation to the evolute.

S. S. G.

X.—ON A PROPERTY OF THE BRACHYSTOCHRONE WHEN THE FORCES ARE ANY WHATEVER.

SIR,—Perhaps you may think fit for insertion the following simple proof, and extension to curves of double curvature, of a Property of the Brachystochrone, demonstrated in Mr. Whewell's *Dynamics*, Part II. sect. 4.

The curve on which a material point will move from one fixed point to another (or from a point to a curve, &c.) in the shortest time under the action of known forces, is such, that the parts of the pressure on it arising from those forces and from the centrifugal force are equal.

Since $t = \int \frac{ds}{v}$, we must have $\delta t = \int \frac{v\delta ds - ds\,\delta v}{v^2} = 0$ between the given limits.

$$\text{Now, } ds^2 = dx^2 + dy^2 + dz^2;$$

$$\text{therefore } \frac{ds}{dt}\,\delta ds = \frac{dx}{dt}\,\delta dx + \frac{dy}{dt}\,\delta dy + \frac{dz}{dt}\,\delta dz;$$

and if XYZ be the components of the accelerating forces acting on the point, then

$$v^2 = \text{const.} + 2\int (\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

$$\text{Hence } v\delta v = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z,$$

$$\text{and } ds\,\delta v = (\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z)\,dt;$$

on substituting these values, we have

$$\delta t = \int \frac{1}{v^2}\left(\frac{dx}{dt}\,d\delta x + \frac{dy}{dt}\,d\delta y + \frac{dz}{dt}\,d\delta z\right) - \int dt\,\frac{1}{v^2}(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) = 0,$$

$$\text{or } \delta t = \frac{1}{v^2}\left(\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y + \frac{dz}{dt}\,\delta z\right)$$

$$- \int dt\left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right)\delta x + \frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right)\delta y + \frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right)dz\right\}$$

$$- \int dt\,\frac{1}{v^2}(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) = 0,$$

by integrating the first term by parts.

Now, in order that the part under the sign of integration may vanish, the coefficients of the variations must separately equal zero, there being no connexion between them. Hence

$$\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{\mathrm{X}}{v^2} = 0,\ \ \frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right) + \frac{\mathrm{Y}}{v^2} = 0,\ \ \frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right) + \frac{\mathrm{Z}}{v^2} = 0;$$

or changing the independent variable,

$$v^2\frac{d^2x}{ds^2} - \frac{dx}{ds}\,v\,\frac{dv}{ds} + \mathrm{X} = 0,\ \ v^2\frac{d^2y}{ds^2} - \frac{dy}{ds}\,v\,\frac{dv}{ds} + \mathrm{Y} = 0,$$

$$v^2\frac{d^2z}{ds^2} - \frac{dz}{ds}\,v\,\frac{dv}{ds} + \mathrm{Z} = 0.$$

If we now multiply these equations by

$$\frac{d^2x}{ds^2},\ \frac{d^2y}{ds^2},\ \frac{d^2z}{ds^2},$$

and add the results, (remembering that since

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1,\ \ \frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0,)$$

we find

$$\frac{v^2}{\rho} = -\left(\mathrm{X}\rho\frac{d^2x}{ds^2} + \mathrm{Y}\rho\frac{d^2y}{ds^2} + \mathrm{Z}\rho\frac{d^2z}{ds^2}\right),$$

ρ being the radius of curvature of the curve at the point xyz.

This result expresses the property above enunciated: for the equations of motion of the point may be put in the form

$$\frac{ds^2}{dt^2}\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2s}{dt^2} = \mathrm{X} + n\rho\frac{d^2x}{ds^2},\ \&\text{c.}$$

where n is the pressure on the curve, or its reaction, which acts in the direction of the radius of curvature.

If we multiply by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ respectively, and add, we get the expression above used for the velocity; but if we multiply them by $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$, $\rho\frac{d^2z}{ds^2}$, and add, we find

$$R = \frac{v^2}{\rho} - \left(X\rho\frac{d^2x}{ds^2} + Y\rho\frac{d^2z}{ds^2} + Z\rho\frac{d^2z}{ds^2}\right).$$

If the curve is restricted to be drawn on a curve surface, whose equation is $L = 0$, the variations are connected by the equation

$$\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z = 0,$$

and we get two equations, from which it is impossible te deduce the property analogous to the above.

If the curve lie in one plane (as that of xy), and the only force act parallel to the axis of x, then the second equation of condition becomes $\frac{1}{v}\frac{dy}{ds} = \text{const.}$ If the force be that of gravity then

$$v = \sqrt{2gx}, \text{ and } \frac{dy}{ds} = \text{const.} \sqrt{2gx} = \sqrt{\frac{x}{2a}} \text{ suppose:}$$

$$\text{therefore } \frac{dx^2}{dy^2} = \frac{ds^2}{dy^2} - 1 = \frac{2a - x}{x},$$

$$\text{and } \frac{dy}{dx} = \frac{x}{\sqrt{2ax - x^2}} = \frac{a}{\sqrt{2ax - x^2}} - \frac{a - x}{\sqrt{2ax - x^2}};$$

$$\text{therefore } y = a \text{ vers.}^{-1}\frac{x}{a} - \sqrt{2ax - x^2},$$

and the brachystochrone is in this case a common cycloid.

I remain, Sir, yours, &c.

Oxford, Oct. 17, 1839. D.

XI.—MATHEMATICAL NOTES.

1. GIVEN the n^{th} part of a straight line, to find the $(n + 1)^{\text{th}}$ part.

Let AP (fig. 8.) be the n^{th} part of AB. Upon AB describe a square, and draw the diagonal AD; join PC, and through E draw FEQ parallel to AC or BD. AQ will be the $(n + 1)^{\text{th}}$ part of

AB. For by similar triangles EQ is to EF as AP to CD. But AP is the n^{th} part of CD; therefore EQ is the n^{th} part of EF, *i.e.* (since AQ = EQ and EF = FD = QB), AQ is the n^{th} part of QB, and therefore the $(n+1)^{\text{th}}$ part of AB. So by joining QC we may find the $(n+2)^{\text{th}}$ part, and so on successively.

This very simple problem is given by Meibomius in the Preface to his edition of *Aristides Quintilianus*, having been suggested, he says, by a figure called Helicon, and used by the ancient writers on harmonics.

2. *Property of the Parabola.*—In Vol. I. p. 205, there were found for the coordinates of the point of intersection of two tangents to a parabola, the expressions

$$y = m(a + a'), \quad x = m\,a\,a',$$

a, a' being the tangents of the angles which the tangents to the curve make with the axis of y. From these expressions it follows, that if y_1, y_2, &c. x_1, x_2, &c. be the coordinates of the angles of any re-entering polygon of $2n$ sides circumscribing a parabola,

$$y_1 - y_2 + y_3, \text{ \&c. } - y_{2n} = 0,$$

$$\text{and } \frac{x_1 x_3 \; . \; . \; . \; . \; x_{2n-1}}{x_2 x_n \; . \; . \; . \; . \; x_{2n}} = 1.$$

Also, the continued product of the abscissæ of the points of intersection of any number of tangents, is equal to the continued product of the abscissæ of the points of contact, provided no three points of intersection lie in the same straight line.

Let x', x'', x''', &c. be the abscissæ of the points of contact, then it is easily seen, from the equation to the parabola, that

$$x' = ma'^2, \quad x'' = ma''^2, \quad x''' = ma'''^2, \text{ \&c.}$$

the continued product of which is

$$x'x''x''' \; . \; . \; . \; x^{(n)} = m^n a'^2 a''^2 a'''^2 \; . \; . \; . \; a^{(n)2}.$$

And if x_1, x_2, x_3, &c. be the coordinates of the points of intersection of the tangents, we have

$$x_1 = ma'a'', \quad x_2 = ma''a''', \quad x_3 = ma''a''', \text{ \&c.}$$

the continued product of which is

$$x_1 x_2 x_3 \; . \; . \; . \; x_n = m^n \, . \, a'^2 a''^2 a'''^2 \; . \; . \; . \; a^{(n)2},$$

which is equal to the preceding expression. It is necessary to limit the intersections in such a way that no three shall lie in the same line, because otherwise some one of the a's in the second series would appear more than twice.

ε.

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I.—SINGULAR SOLUTIONS AND PARTICULAR INTEGRALS OF DIFFERENTIAL EQUATIONS.

(*Continued.*)

By W. WALTON, B.A., Trinity College.

1. LET $c = wv^a + u$ be a regular integral of a differential equation of any order between two variables, and let $v = 0$ be a singular solution. Then differentiating with respect to x, and multiplying the result by v^{1-a}, we get

$$0 = \left(\frac{dw}{dx}\right) v + aw \left(\frac{dv}{dx}\right) + v^{1-a} \left(\frac{du}{dx}\right) = \text{V suppose.}$$

Let y_m represent the m^{th} differential coefficient of y with respect to x. Then manifestly, since a is a positive quantity less than unity, and since $\left(\frac{du}{dx}\right)$ is necessarily neither zero nor infinity for all the simultaneous values of the variables defined by the equation $v = 0$, we shall have, under the condition $v = 0$,

$\frac{dV}{dy_m} = 0$, if v involve neither y_m nor y_{m-1}.

$\frac{dV}{dy_m} = aw \frac{dv}{dy_{m-1}}$, a finite quantity, if v involve y_{m-1} and not y_m.

$\frac{dV}{dy_m} = \infty$, if v involve y_m.

Ex. Let us take the equation

$$0 = 3y_2(xy + a) + y_1(y + xy_1) + 3y_3(xy + a)^{\frac{2}{3}} =$$

which is derived, according to the above method, from the equation

$$c = y_1 (xy + a)^{\frac{1}{3}} + y_2.$$

Then clearly, under the condition $xy + a = 0$, we have

$$\frac{dV}{dy_2} = 3(xy + a) = 0,$$

$$\frac{dV}{dy_3} = 3(xy + a)^{\frac{2}{3}} = 0,$$

$$\frac{dV}{dy_1} = y + xy_1 + xy_1 = xy_1,$$

$$\frac{dV}{dy} = 3xy_2 + xy_1^2 + 2y_3x(xy + a)^{\frac{1}{3}} = \infty,$$

$$\frac{dV}{dx} = 3yy_2 + y_1^2 + 2yy_3(xy + a)^{-\frac{1}{3}} = \infty.$$

2. Let y_r and y_s represent the r^{th} and the s^{th} differential coefficients of y. Then, differentiating the equation $V = 0$ on the hypothesis that y_r varies in consequence of the variation of y_s independently of the other variables, we get

$$\frac{dV}{dy_s} + \frac{dV}{dy_r}\frac{dy_r}{dy_s} = 0 \quad \ldots\ldots\ldots\ldots (k).$$

1st. Suppose that v involves y_s and not y_r. Then, since by the preceding article, under the condition $v = 0$, we have

$$\frac{dV}{dy_s} = \infty,$$

$$\text{and } \frac{dV}{dy_r} = \text{zero or a finite quantity.}$$

It is clear, that under these circumstances

$$\frac{dy_r}{dy_s} = \infty.$$

2nd. Suppose that v involves y_{s-1}, and none of the quantities y_s, y_{r-1}, y_r; then, since by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity,}$$

$$\text{and } \frac{dV}{dy_r} = 0.$$

It is clear from the equation (k), that under these circumstances

$$\frac{dy_r}{dy_s} = \infty.$$

Ex. Let $c = y_3(xy + y_1)^{\frac{1}{3}} + y_2$ be a first integral of a differential equation. The differential equation derived from it, as in Art. (1), will be

$$0 = 3y_4(xy + y_1) + y_3(y + xy_1 + y_2) + 3y_3(xy + y_1)^{\frac{2}{3}}.$$

Differentiating as if y_4 and y_1 alone varied, we have

$$0 = 3\frac{dy_4}{dy_1}(xy + y_1) + 3y_4 + xy_3 + 2y_3(xy + y_1)^{-\frac{1}{3}};$$

and therefore when we put $xy + y_1 = 0$, we have evidently

$$\frac{dy_4}{dy_1} = x.$$

In precisely the same way we should find, under the condition $xy + y_1 = 0$, the following relations:

$$\frac{dy_4}{dx} = x, \quad \frac{dy_4}{dy} = \infty,$$

$$\frac{dy_3}{dx} = x, \quad \frac{dy_3}{dy} = \infty, \quad \frac{dy_3}{dy_1} = \infty,$$

$$\frac{dy_2}{dx} = \infty, \quad \frac{dy_2}{dy} = x, \quad \frac{dy_2}{dy_1} = \infty.$$

Again, to afford an illustration of the second case mentioned in this article, we will differentiate as if y_2 and y_3 alone varied, and thus we have

$$0 = \frac{dy_3}{dy_2}(y + xy_1 + y_2) + y_3 + 3\frac{dy_3}{dy_2}(xy + y_1)^{\frac{2}{3}};$$

and therefore, evidently when we put $xy + y_1 = 0$, we have

$$\frac{dy_3}{dy_2} = \infty.$$

In the same way we may shew, that under the condition $xy + y_1 = 0$,

$$\frac{dy_4}{dy_2} = \infty.$$

3. Let the equation $V = 0$ be of the n^{th} order, and the singular solution $v = 0$ of the τ^{th} order. Then clearly, under the general results of the preceding article, are comprehended the series of relations

$$\frac{dy_{\tau+1}}{dy_\tau} = \infty, \quad \frac{dy_{\tau+2}}{dy_\tau} = \infty, \quad \frac{dy_{\tau+3}}{dy_\tau} = \infty, \ldots\ldots \frac{dy_n}{dy_\tau} = \infty,$$

$$\text{and } \frac{dy_{\tau+2}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+3}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+4}}{dy_{\tau+1}} = \infty, \ldots\ldots \frac{dy_n}{dy_{\tau+1}} = \infty.$$

4. Let $W = 0$ represent the equation $V = 0$ under any state of

modification. Then, differentiating as in the equation (k) of Art. 2, we have

$$\frac{dW}{dy_s} + \frac{dW}{dy_r}\frac{dy_r}{dy_s} = 0 \quad \ldots\ldots\ldots (g).$$

but under the circumstances of Art. (2) we have

$$\frac{dy_r}{dy_s} = \infty,$$

and therefore by equation (g) we must have

$$\frac{dW}{dy_s} : \frac{dW}{dy_r} = \infty.$$

5. Let $U = 0$ represent the equation $V = 0$, cleared of radicals. Then, by the preceding article, and under the circumstances of Art. (2), we must have

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = \infty.$$

But $\frac{dU}{dy_s}$ cannot assume an infinite value when submitted to the relation $v = 0$, since U, and therefore $\frac{dU}{dy_s}$, does not involve radicals. Hence clearly, we must have

$$\frac{dU}{dy_r} = 0.$$

Ex. Take the equation

$$0 = (1 + y_3)^2 - 4y_1^2 . (x + y_2) = U,$$

which has $x + y_2 = 0$ as a singular solution.

Then clearly, under the condition $x + y_2 = 0$, we have

$$\frac{dU}{dy} = 0,$$

$$\frac{dU}{dy_1} = -8y_1(x + y_2) = 0,$$

$$\frac{dU}{dy_3} = 2(1 + y_3) = 0.$$

6. Let the equation $U = 0$ be of the n^{th} order, and the singular solution $v = 0$ of the r^{th} order. Then clearly, under the general results of the preceding article, are comprehended the relations

$$\frac{dU}{dy_{r+1}} = 0, \quad \frac{dU}{dy_{r+2}} = 0, \quad \frac{dU}{dy_{r+3}} = 0, \ldots\ldots \frac{dU}{dy_n} = 0,$$

7. Let $v = 0$ be a particular integral corresponding to an infinite value of c. Then putting $-a$ in place of a, we have

$$V = \left(\frac{dw}{dx}\right) v - aw\left(\frac{dv}{dx}\right) + v^{1+a}\left(\frac{du}{dx}\right) = 0.$$

Now the most general form for w is

$$\rho_0 + \rho_1 v^{\beta_1} + \rho_2 v^{\beta_2} + \ldots\ldots$$

where $\beta_1, \beta_2, \beta_3, \ldots\ldots$ are all positive quantities. Hence

$$\left(\frac{dw}{dx}\right) v = \left(\frac{d\rho_0}{dx}\right) v + \left(\frac{d\rho_1}{dx}\right) v^{1+\beta_1} + \left(\frac{d\rho_2}{dx}\right) v^{1+\beta_2} + \ldots\ldots$$

$$+ (\beta_1\rho_1 v^{\beta_1} + \beta_2\rho_2 v^{\beta_2} + \ldots\ldots) \left(\frac{dv}{dx}\right);$$

and therefore we have

$$V = v^{1+a}\left(\frac{du}{dx}\right) + \left(\frac{d\rho_0}{dx}\right) v + \left(\frac{d\rho_1}{dx}\right) v^{1+\beta_1} + \left(\frac{d\rho_2}{dx}\right) v^{1+\beta_2} + \ldots$$

$$+ \{-a\rho_0 + (\beta_1 - a)\rho_1 v^{\beta_1} + (\beta_2 - a)\rho_2 v^{\beta_2} + \ldots\}\left(\frac{dv}{dx}\right) = 0 \ldots (\lambda).$$

Hence if we put $v = 0$ in the expression for $\dfrac{dV}{dy_m}$, we shall have

$\dfrac{dV}{dy_m} = 0$, if v involve neither y_m nor y_{m-1},

$\dfrac{dV}{dy_m} = -a\rho_0 \dfrac{dv}{dy_{m-1}}$, a finite quantity, if v involve y_{m-1} and not y_m,

and, if v involve y_m,

$$\frac{dV}{dy_m} = \{\beta_1\rho_1(\beta_1 - a) v^{\beta_1 - 1} + \beta_2\rho_2(\beta_2 - a) v^{\beta_2 - 1} + \ldots\} \frac{dv}{dy_m}\left(\frac{dv}{dx}\right)$$

$$+ \left(\frac{d\rho_0}{dx}\right)\frac{dv}{dy_m} - a\rho_0 \frac{d}{dy_m}\left(\frac{dv}{dx}\right)$$

$$= \left(\frac{d\rho_0}{dx}\right)\frac{dv}{dy_m} - a\rho_0\frac{d}{dy_m}\left(\frac{dv}{dx}\right),$$

since the former part of the expression vanishes by virtue of the equation (λ).

It is also easily seen, by the aid of the equation (λ), that

$$\frac{d}{dy_m}\left(\frac{dv}{dx}\right) = 0 \text{ if } \rho_0 \text{ be a constant quantity,}$$

and that it remains finite if ρ_0 be variable. Hence

$$\frac{dV}{dy_m} = 0 \text{ if } \rho_0 \text{ be a constant quantity,}$$

and a finite quantity, if ρ_0 be variable.

Ex. Take the equation

$$0 = 2y_2(x + y_2) - y_1(1 + y_3) + 2y_1(x + y_2)^{\frac{3}{2}} = V,$$

of which $x + y_2 = 0$ is a particular integral corresponding to an infinite value of the arbitrary constant in the integral

$$c = y_1(x + y_2)^{-\frac{1}{2}} + y.$$

Then clearly, under the condition $x + y_2 = 0$, we have

$$\frac{dV}{dy_1} = -(1 + y_3) + 2(x + y_2)^{\frac{3}{2}} = 0,$$

$$\frac{dV}{dy_3} = -y_1.$$

$$\frac{dV}{dy_2} = 2(x + y_2) + 2y_2 + 3y_1(x + y_2)^{\frac{1}{2}} = 2y_2.$$

Ex. Take the equation

$$0 = -a(1 + y_3) + 2y_1(x + y_2)^{\frac{3}{2}} = V,$$

of which $x + y_2 = 0$ is a first integral corresponding to an infinite value of the arbitrary constant. Then, under the condition $x + y_2 = 0$, we have

$$\frac{dV}{dy_2} = 3y_1(x + y_2)^{\frac{1}{2}} = 0,$$

$$\frac{dV}{dx} = 3y_1(x + y_2)^{\frac{1}{2}} = 0.$$

Ex. Take the equation

$$0 = x + y_2 - x(1 + y_3) - \tfrac{1}{2}(1 + y_3)(x + y_2)^{\frac{1}{2}} + y_1(x + y_2)^2 = V,$$

of which $c = \{x + (x + y_2)^{\frac{1}{2}}\}(x + y_2)^{-1} + y$ is a regular integral, and $x + y_2 = 0$ a particular integral for $c = \infty$.

Differentiating, we have

$$\frac{dV}{dy_2} = 1 - \tfrac{1}{4}(1 + y_3)(x + y_2)^{-\frac{1}{2}} + 2y_1(x + y_2);$$

but from the equation $V = 0$, we have

$$1 + y_3 = \frac{x + y_2 + y_1(x + y_2)^2}{x + \frac{1}{2}(x + y_2)^{\frac{1}{2}}},$$

and therefore

$$\frac{dV}{dy_2} = 1 - \frac{(x + y_2)^{\frac{1}{2}} + y_1(x + y_2)^{\frac{3}{2}}}{4x + 2(x + y_2)^{\frac{1}{2}}} + 2y_1(x + y_2)$$

$= 1$ under the condition $x + y_2 = 0$.

If we had taken the differential equation corresponding to a regular integral

$$c = \{a + (x + y_2)^{\frac{1}{2}}\}(x + y_2)^{-1} + y,$$

we should have got under the condition $x + y_2 = 0$,

$$\frac{dV}{dy_2} = 0.$$

8. Differentiating the equation $V = 0$ on the hypothesis that y,

varies in consequence of the variation of y_s, independently of the other variables, we get

$$\frac{dV}{dy_s} + \frac{dV}{dy_r}\frac{dy_r}{dy_s} = 0 \;\ldots\ldots\ldots\ldots\; (m).$$

1st. Suppose that v involves y_{s-1}, and none of the quantities y_s, y_{r-1}, y_r.

Then since, by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity},$$

$$\text{and} \quad \frac{dV}{dy_r} = 0,$$

we have, by the equation (m) under these circumstances,

$$\frac{dy_r}{dy_s} = \infty.$$

2nd. Suppose that v involves y_s and y_{r-1}, and not y_r. Then since by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity or zero},$$

$$\text{and} \quad \frac{dV}{dy_r} = \text{a finite quantity},$$

we have by the equation (m) under these circumstances,

$$\frac{dy_r}{dy_s} = \text{a finite quantity or zero}.$$

We may obviously obtain, under the condition $v=0$, for different values of r and s a variety of such relations as these.

We will furnish the following exemplifications of the two classes of relations at which we have arrived.

Ex. Take the equation

$$0 = y_3(xy + y_1) - y_2(y + xy_1 + y_2) + y_4(xy + y_1)^2,$$

of which $xy + y_1 = 0$ is a particular integral corresponding to an infinite value of the arbitrary constant of a first integral.

Differentiating as if y_2 and y_4 alone varied, we have

$$0 = -(y + xy_1 + y_2) - y_2 + \frac{dy_4}{dy_2}(xy + y_1)^2;$$

and therefore, under the condition $xy + y_1 = 0$, we clearly have

$$\frac{dy_4}{dy_2} = \infty.$$

Again, differentiating as if y and y_2 alone varied, we have

$$0 = xy_3 - (y + xy_1 + y_2)\frac{dy_2}{dy} - y_2\left(1 + \frac{dy_2}{dy}\right) + 2xy_4(xy + y_1);$$

and therefore, under the condition $xy + y_1 = 0$, we have

$$\frac{dy_2}{dy} = \frac{y_2 - xy_3}{y_2}.$$

Ex. Take the equation

$$0 = -b(y_3 + y_1) - \tfrac{2}{3}(y_3 + y_1)(y_2 + y)^{\frac{3}{2}} + y_1(y_2 + y)^2,$$

of which $c = \{b + (y_2 + y)^{\frac{1}{2}}\}(y_2 + y)^{-1} + y$ is a first integral.

Then differentiating as if y and y_3 alone varied, we have

$$0 = -b\frac{dy_3}{dy} - \tfrac{2}{3}\frac{dy_3}{dy}(y_2 + y)^{\frac{3}{2}} - 3(y_3 + y_1)(y_2 + y)^{\frac{1}{2}} + 2y_1(y_2 + y),$$

and therefore if we put $y_2 + y = 0$, we clearly have

$$\frac{dy_3}{dy} = 0.$$

9. Let the equation $V = 0$ be of the n^{th} order, and the particular integral $v = 0$ of the r^{th} order. Then, under the condition $v = 0$, the general results of Art. (8) comprehend the following relations:

$$\frac{dy_{r+2}}{dy_r} = \infty, \quad \frac{dy_{r+3}}{dy_r} = \infty, \ldots\ldots \frac{dy_n}{dy_r} = \infty.$$

10. Let $U = 0$ represent the equation $V = 0$ cleared of radicals. Then differentiating on the same hypothesis as in the equation (m), we have

$$\frac{dU}{dy_s} + \frac{dU}{dy_r}\frac{dy_r}{dy_s} = 0 \ldots\ldots\ldots (w).$$

1st. Suppose that v involves y_{s-1}, and none of the three quantities y_s, y_{r-1}, y_r. Then since by Art. 8, under the condition $v = 0$, we have

$$\frac{dy_r}{dy_s} = \infty,$$

it is clear by the equation (w) that

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = \infty;$$

but $\frac{dU}{dy_s}$ cannot assume an infinite value, since U, and therefore $\frac{dU}{dy_s}$ involves no radicals. Hence clearly

$$\frac{dU}{dy_r} = 0.$$

2nd. Suppose that v involves y_s, and y_{r-1}, and not y_r. Then since by Art. 8, under the condition $v = 0$,

$$\frac{dy_r}{dy_s} = \text{a finite quantity or zero},$$

we see by the equation (w), that

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = 0, \text{ or a finite quantity,}$$

and therefore $\dfrac{dU}{dy_s} = 0$, or a finite quantity.

Ex. Take the equation

$$0 = \{a(1+y_2) - y_3(x+y_1)^2\}^2 - \tfrac{1}{4}(x+y_1)(1+y_2)^2 = U,$$

of which $c = \{a + (x+y_1)^{\frac{1}{2}}\}(x+y_1)^{-1} + y_2$ is a first integral.

Then

$$\frac{dU}{dy_3} = -2\{a(1+y_2) - y_3(x+y_1)^2\}(x+y_1)^2 = 0, \text{ putting } x+y_1=0;$$

also,

$$\frac{dU}{dy_1} = -4\{a(1+y_2) - y_3(x+y_1)^2\}y_3(x+y_1) - \tfrac{1}{4}(1+y_2)^2.$$

Hence, $\dfrac{dU}{dy_1} : \dfrac{dU}{dy_3} = \infty$, when we put $x + y_1 = 0$, as is readily seen when we substitute for $1 + y_2$ its value, as derived from the equation $U = 0$.

11. Let the equation $U = 0$ be of the n^{th} order, and $v = 0$ of the τ^{th}. Then, clearly, the general results of Art. 10 comprehend the following relations:

$$\frac{dU}{dy_{\tau+2}} = 0, \quad \frac{dU}{dy_{\tau+3}} = 0, \quad \frac{dU}{dy_{\tau+4}} = 0, \ldots\ldots \frac{dU}{dy_n} = 0.$$

12. If $v = 0$ had been a particular integral corresponding to a finite value of the arbitrary constant, we should have had

$$V = \left(\frac{dw}{dx}\right) v + aw\left(\frac{dv}{dx}\right) = 0,$$

instead of the expression of Art. 7. And it will be readily seen, on reverting to the analysis, that we should have arrived at precisely the same results for this as for the former expression for V.

13. Let $v = 0$ be a solution of the equation $V = 0$, and let $v = 0$ be of the τ^{th} and $V = 0$ of the n^{th} order. Suppose that $\dfrac{dy_{\tau+1}}{dy_\tau} = \infty$, under the condition $v=0$. Then $v = 0$ cannot be a particular integral, since, as we have shewn in Art. 8, $\dfrac{dy_{\tau+1}}{dy_\tau}$ would, in this case, have been equal to zero, or to a finite quantity. Hence, manifestly the following method will enable us always to determine all the singular solutions of a differential equation of any order.

Assume

$$\frac{dy_1}{dy} = \infty, \frac{dy_2}{dy_1} = \infty, \frac{dy_3}{dy_2} = \infty, \ldots\ldots \frac{dy_n}{dy_{n-1}} = \infty,$$

and let

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 0, \ldots\ldots v_{n-1} = 0,$$

be equations which satisfy respectively the preceding relations, the subscript numbers being characteristic of their respective orders. Then, as many of these equations as satisfy the equation $V = 0$ are singular solutions.

Again, suppose that $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}} = \infty$ under the condition $v = 0$, $U = 0$ representing the equation $V = 0$, cleared of radicals. Then $v = 0$ cannot be a particular integral, since, as we have shewn in Art. 10, $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}}$ would in this case have been equal to zero or to a finite quantity. Hence, obviously, we may ascertain a singular solution of any assigned order of a differential equation by the following method:

Assume

$$\frac{dU}{dy_{r+1}} = 0, \frac{dU}{dy_{r+2}} = 0, \frac{dU}{dy_{r+3}} = 0, \ldots\ldots \frac{dU}{dy_n} = 0,$$

and eliminate $y_{r+1}, y_{r+2}, y_{r+3}, \ldots\ldots\ldots\ldots y_n$ between these equations and the equation $U = 0$. Then, if the resulting equation satisfy the relation $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}} = \infty$, it will be a singular solution.

14. Let $v = 0$ be a solution of the equation $V = 0$, and let $v = 0$ be of the r^{th} and $V = 0$ of the n^{th} order. Suppose that $\frac{dy_{r+2}}{dy_r} = \infty$, and $\frac{dy_{r+1}}{dy_r} =$ zero, or a finite quantity, under the condition $v = 0$. Then $v = 0$ cannot be a singular solution, since, as we have shewn in Art. 3, $\frac{dy_{r+1}}{dy_r}$ would in this case have been equal to infinity. Hence, by the following method, we may always ascertain all the particular integrals of every order of any differential equation.

Assume

$$\frac{dy_2}{dy} = \infty, \frac{dy_3}{dy_1} = \infty, \frac{dy_4}{dy_2} = \infty, \ldots\ldots \frac{dy_n}{dy_{n-2}} = \infty,$$

and let

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 0 \ldots\ldots v_{n-2} = 0,$$

be equations which satisfy respectively the preceding relations, the subscript numbers being characteristic of their respective orders. Then, as many of these equations as satisfy the equation $V = 0$, and dissatisfy respectively the relations

$$\frac{dy_1}{dy} = \infty, \quad \frac{dy_2}{dy_1} = \infty, \quad \frac{dy_3}{dy_2} = \infty, \ \ldots\ldots \frac{dy_{n-1}}{dy_{n-2}},$$

are particular integrals of the equation $V = 0$.

Again, suppose that $\dfrac{dU}{dy_{r+1}} \ \dfrac{dU}{dy_{r+2}} = \infty$, and $\dfrac{dU}{dy_r} : \dfrac{dU}{dy_{r+1}} =$ zero or a finite quantity, $U = 0$ representing the equation $V = 0$ cleared of radicals. Then $v = 0$ cannot be a singular solution, since, as we have shewn in Art. 5, $\dfrac{dU}{dy_r} : \dfrac{dU}{dy_{r+1}}$ would in this case be equal to infinity. Hence we may obviously ascertain all the particular integrals of any assigned order of which a differential equation of any order is susceptible, by the following method:

Assume

$$\frac{dU}{dy_{r+2}} = 0, \quad \frac{dU}{dy_{r+3}} = 0, \quad \frac{dU}{dy_{r+4}} = 0, \ \ldots\ldots \frac{dU}{dy_n} = 0,$$

and eliminate between these relations and the equation $U = 0$, as many of the quantities y_{r+1}, y_{r+2}, ... y_n as possible, that is, all but one. Then, if the resulting equation be susceptible of satisfaction by an equation $v = 0$ of the r^{th} order, $v = 0$ will be a particular integral of $U = 0$, provided that it do not make

$$\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}} = \infty.$$

It is worthy of observation, that the above processes enable us to determine only those particular integrals of a differential equation, which are of an order at least two dimensions lower than the proposed equation.

15. Let $c = wv^a + u$ denote a regular integral of a differential equation, and let $P = 0$ denote the same equation cleared of radicals and of fractions. Then differentiating the equation $P = 0$ as if c varied in consequence of the variation of y_m independently of the other variables, we have

$$\frac{dP}{dy_m} + \frac{dP}{dc}\frac{dc}{dy_m} = 0.$$

1st. Let a be a positive quantity less than unity, and let v involve y_m. Then clearly, under the condition $v = 0$, we have

$$\frac{dc}{dy_m} = \infty,$$

and therefore

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

for a value u of c, that is, for a value of c corresponding to a singular solution if u be a variable quantity, or a particular integral if u be constant. But evidently $\dfrac{dP}{dy_m}$ cannot be equal to infinity

when we put $v = 0$ or $c = u$, because P involves neither fractions nor radicals. Hence we must have for $c = u$

$$\frac{dP}{dc} = 0.$$

2nd. Let a be a negative quantity, and let v involve y_m. Then clearly, under the condition $v = 0$,

$$\frac{dc}{dy_m} = \infty,$$

and therefore

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

for a value ∞ of c. But it is clear from the nature of P that we cannot have, as in the first case, for the value of c the relation

$$\frac{dP}{dc} = 0.$$

3rd. Let a be a positive quantity greater than or equal to unity, and let v involve y_m. Then clearly, under the condition $v = 0$,

$$\frac{dc}{dy_m} = \text{zero, or a finite quantity,}$$

$$\text{and therefore } \frac{dP}{dy_m} : \frac{dP}{dc} = \text{zero, or a finite quantity,}$$

for a value of c corresponding either to no solution at all of the differential equation, u being a variable quantity, or to a particular integral, u being a constant quantity.

Hence from what has been said we are enabled to avail ourselves of the following process for the determination of the singular solutions of differential equations from their regular integrals.

Let $P = 0$ represent a regular integral cleared of radicals and of fractions, the arbitrary constant c being involved in P.

Assume $\frac{dP}{dc} = 0$, and let u be a variable quantity deducible from this equation as a value for c. Substitute this value of c in the expression P, and let the result be represented by P′. Then if P′ have any factor v such that, for any one of the quantities x, y, y_1, y_2, ... which v involves, when $v = 0$, and $P = 0$,

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

$v = 0$ will be a singular solution of the differential equation belonging to $P = 0$ as a regular integral.

And if we obtain every such factor of P′, and equate each of them to zero, we shall obtain all the possible singular solutions for the value u of c.

If there be more variable values of c than one deducible from the equation $\frac{dP}{dc} = 0$, we must proceed in the same way with each of them; and ultimately we shall have obtained all the possible singular solutions.

Ex. Let $c = x^2y(x+y)^{\frac{1}{3}} + x$ be the complete primitive of an equation of the first order. Clearing it of radicals, we have

$$P = (c-x)^3 - x^6y^3(x+y) = 0 \text{ ...... } (f),$$

$$\frac{dP}{dc} = 3(c-x)^2 = 0 \text{ suppose;}$$

$$\therefore c = x;$$

$$\therefore P' = -x^6y^3(x+y) = 0,$$

and the factors of P' are x, y, and $x+y$.

$$\frac{\frac{dP}{dx}}{\frac{dP}{dc}} = \frac{-3(c-x)^2 - 6x^5y^3(x+y) - x^6y^3}{3(c-x)^2},$$

$$= -1 - \tfrac{1}{3}xy \cdot \frac{5x+6y}{(x+y)^{\frac{2}{3}}}, \text{ from the equation } (f),$$

$$= -1, \text{ when we put } x = 0;$$

hence $x = 0$ is not a singular solution.

Again, $$\frac{\frac{dP}{dy}}{\frac{dP}{dc}} = \frac{-3x^6y^2(x+y) - x^6y^3}{3(c-x)^2},$$

$$= \frac{-3x^6y^2(x+y) - x^6y^3}{3x^4y^2(x+y)^{\frac{2}{3}}},$$

$$= \frac{-3x^2(x+y) - x^2y}{3(x+y)^{\frac{2}{3}}}$$

$$= -x^{\frac{4}{3}} \text{ when we put } y = 0;$$

hence $y = 0$ is not a singular solution.

Again, it is clear that either $\frac{dP}{dx} : \frac{dP}{dc}$, or $\frac{dP}{dy} : \frac{dP}{dc}$ assumes an infinite value when we put $x + y = 0$, Hence

$$x + y = 0$$

is a singular solution of the differential equation whose complete primitive is

$$(c-x)^3 - x^6y^3(x+y) = 0.$$

If this equation had been presented to us in its original shape,

$$c = x^2 y (x + y)^{\frac{1}{3}} + x,$$

of course we could have seen these conclusions at once.

It would sometimes be very troublesome to clear an equation of radicals. Thus, if

$$c = x (x + y)^{\frac{1}{3}} + y (x^2 + y)^{\frac{1}{2}},$$

we should be going greatly out of our way if we were to endeavour to clear it of radicals, in order to ascertain the singular solutions of the corresponding differential equation.

II.—CASE OF AN APSIDAL ANGLE IN A PATH NEARLY CIRCULAR.

DETERMINATION of the angle between the apsides of the projection upon an horizontal plane of the path of a material particle, moving within a surface of revolution whose axis is vertical, the path of the particle being nearly circular.

Let x, y, z be the coordinates of the body's position at any time t, the axis of the surface being the axis of z. Let ρ denote the distance of the projection of the body upon the plane of xy from the axis of z, and θ the angle between ρ and x.

Then, since the forces which act upon the body have no moments about the axis of z, we have, by the principle of the conservation of areas,

$$\rho^2 \frac{d\theta}{dt} = h \text{ .... } (1),$$

where h is a constant quantity.

Again, since the velocity of the body must be the same as if it had fallen freely from its prime position to its altitude z, we have

$$\frac{d}{dt} \frac{ds^2}{dt^2} = -2g \frac{dz}{dt},$$

$$\text{but } ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2,$$

and therefore

$$\frac{d}{dt} \left(\frac{d\rho^2}{dt^2} + \rho^2 \frac{d\theta^2}{dt^2} + \frac{dz^2}{dt^2} \right) = -2g \frac{dz}{dt},$$

and therefore

$$\frac{d}{d\theta} \left\{ \left(\frac{d\rho^2}{d\theta^2} + \rho^2 + \frac{dz^2}{d\rho^2} \frac{d\rho^2}{d\theta^2} \right) \frac{d\theta^2}{dt^2} \right\} = -2g \frac{dz}{d\rho} \frac{d\rho}{d\theta}.$$

Hence, from the equation (1),

$$\frac{d}{d\theta}\left\{\left(\frac{d\rho^2}{d\theta^2}+\rho^2+\frac{dz^2}{d\rho^2}\frac{d\rho^2}{d\theta^2}\right)\frac{h^2}{\rho^4}\right\}=-2g\frac{dz}{d\rho}\frac{d\rho}{d\theta}.$$

Put $\frac{dz}{d\rho}=p$ and substitute $\frac{1}{u}$ for ρ, and we have

$$\frac{d}{d\theta}\left(\frac{du^2}{d\theta^2}+p^2\frac{du^2}{d\theta^2}+u^2\right)=\frac{2g}{h^2}\frac{p}{u^2}\frac{du}{d\theta};$$

$$\therefore \frac{d^2u}{d\theta^2}(1+p^2)+p\frac{dp}{d\theta}\frac{du}{d\theta}+u=\frac{gp}{h^2u^2}\ \ldots\ldots(2).$$

Let $u=c+w$, c being the value of u at an apse.

Then clearly if $p=p_1$, and $\frac{dp}{d\rho}=q_1$ when $u=c$, we have

$$p=p_1+\frac{dp_1}{dc}w \text{ nearly since } w \text{ is small,}$$

$$=p_1-q_1\frac{w}{c^2}.$$

Hence the equation (2) becomes

$$\frac{d^2w}{d\theta^2}\left\{1+\left(p_1-\frac{q_1w}{c^2}\right)^2\right\}+\left(p_1-\frac{q_1w}{c^2}\right)\frac{q_1}{c^2}\frac{dw^2}{d\theta^2}+c+w=$$

$$=\frac{g}{h^2}\left(\frac{p_1}{u^2}-\frac{q_1w}{c^2u^2}\right);$$

and therefore neglecting squares and products of small quantities, we get

$$\frac{d^2w}{dt^2}(1+p_1^2)+c+w-\frac{g}{h^2}\left(\frac{p_1}{c^2+2cw}-\frac{q_1w}{c^4}\right)=0,$$

$$\therefore \frac{d^2w}{dt^2}(1+p_1^2)+c+w-\frac{g}{c^2h^2}\left\{p_1\left(1-\frac{2w}{c}\right)-\frac{q_1w}{c^2}\right\}=0;$$

$$\therefore \frac{d^2w}{dt^2}(1+p_1^2)+c+w-\frac{gp_1}{c^2h^2}+\frac{gw}{c^2h^2}\left(\frac{2p_1}{c}+\frac{q_1}{c^2}\right)=0\ldots(3).$$

Now when the orbit is a circle with radius equal to $\frac{1}{c}$ we have, omitting the terms depending upon w,

$$c-\frac{gp_1}{h^2c^2}=0,$$

$$\therefore \frac{g}{c^2h^2}=\frac{c}{p_1}.$$

Introducing this relation into the small term in the equation (3), we have

$$(1+p_1^2)\frac{d^2w}{dt^2}+c+w-\frac{gp_1}{c^2h^2}+\frac{cw}{p_1}\left(\frac{2p_1}{c}+\frac{q_1}{c^2}\right)=0;$$

$$\therefore\ (1 + p_1^2)\frac{d^2w}{dt^2} + c - \frac{gp_1}{c^2h^2} + \frac{3cp_1 + q_1}{cp_1}w = 0.$$

And from this equation, by the ordinary method, we get for the angle between the apsides, the expression

$$\pi\sqrt{\frac{cp_1(1 + p_1^2)}{3cp_1 + q_1}},$$

$$\text{or } \pi\sqrt{\frac{p_1(1 + p_1^2)}{3p_1 + q_1 a}}, \text{ if } c = \frac{1}{a}.$$

J. F. H.

III.—RESEARCHES ON THE THEORY OF ANALYTICAL TRANSFORMATIONS, WITH A SPECIAL APPLICATION TO THE REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER.

By G. Boole, Waddington, near Lincoln.

Let P be a function of x and y: then it is clear that, whatever value we give to those variables, (and they are in this instance supposed to be without limitation,) P will assume some corresponding value, real or imaginary. Let us now suppose that x and y bear such relations to two other variable quantities, x' and y', that for every pair of values the former may be supposed to assume, the latter receive corresponding values. This is equivalent to supposing

$$x = f(x', y'), \quad y = f'(x', y') \quad \ldots\ldots\ldots\ldots (1),$$

and does not in any way limit the generality which we suppose x and y to possess.

If we substitute for x and y the values supposed to be given in (1), we shall have the general equation

$$P = P';$$

and this will be true for all supposable values of x and y.

Suppose, now, $P = 0$ to be the equation of a curve, P being, as before, a function of x and y. This equation we may consider under two distinct points of view: first, as expressing a relation between x and y for each point of the curve, which is the ordinary, and I believe hitherto the only method of considering the subject; or, secondly, as expressing a particular state or condition of the function P. Geometrically speaking, this latter view is tantamount to considering any plane curve $\phi(x, y) = 0$, as

formed by the intersection of the surface $\phi(x, y) = z$ with the plane $z = 0$, that is, with the plane x, y.

Let it now be required to transform the equation $P = 0$ into the equation $P' = 0$ by the substitution of the values $x = f(x', y')$, $y = f'(x', y')$.

The order of proceeding it is here important to observe. We should first substitute in the function P the *general* values of x and y, and afterwards introduce the particular condition $P = 0$. The transformation of P into P′ is therefore independent of any relation among the variables supposed to be expressed in the condition $P = 0$. We may, therefore, by the reasoning of the preceding section, make $P = P'$, whether the values attributed to x and y satisfy the primitive equation to the curve or not. The same remark may be made respecting the various orders of differentials; we shall therefore have the following system of equations universally true:

$$P = P',$$
$$\frac{dP}{dx} = \frac{dP'}{dx} = \frac{dP'}{dx'}\frac{dx'}{dx} + \frac{dP'}{dy'}\frac{dy'}{dx},$$
$$\frac{dP}{dy} = \frac{dP'}{dy} = \frac{dP'}{dx'}\frac{dx'}{dy} + \frac{dP'}{dy'}\frac{dy'}{dy},$$
&c. &c.

In applying these principles to the transformation of any particular equation, we are at liberty, after performing the requisite differentiations, to replace the primitive condition $P = 0$ by any other which the nature of the problem may render it advisable to introduce. Interpreted into geometrical language, the above implies, that if any curve line in the plane xy be considered as formed by the intersection of the surface whose equation is

$$\phi(x, y) = z$$

with that plane; and if the co-ordinates x, y be transferred into another system x', y', not only will the line of intersection continue the same as before, but the intersecting surface also, throughout its whole extent.

In making use of the differential equations of the first or higher orders,

$$\frac{dP}{dx} = \frac{dP'}{dx}, \qquad \frac{dP}{dy} = \frac{dP'}{dy},$$
$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \qquad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy},$$
&c. &c.

we must be careful to introduce new conditions only after performing the differentiations, or in such a way as to produce the same result. If, for example, we have in P′ a term y'^2, and suppose $y' = \phi(x, y) = 0$, we may in the first differentiatior such a term, because y' will be retained as a coefficient

cannot do this in taking the partial differentials of the second order which do not contain y'. Although the sum of the partial differentials of any order is zero, yet those partial differentials themselves will be susceptible of real values, which it will be necessary to take account of.

As a first application of the preceding theory, let it be required to transform the general equation of the second order for two variables,

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = 0,$$

into a new equation with rectangular co-ordinates x', y', and of the form

$$A_1x'^2 + A_1'y'^2 + D_1 = 0.$$

By the reasoning of the preceding sections, the first members of these two equations must be equal for all values of x and y. As we are at liberty to introduce a new condition, and as one object to be determined is the inclination of the axes x and x', let us assume $y' = 0$; then will $x'^2 = (x - a)^2 + (y - b)^2$, making a and b the co-ordinates of the new centre.

The equation $P = P'$ gives

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = A_1x'^2 + A_1'y'^2 + D_1 \ \ldots (1).$$

The equation $\dfrac{dP}{dx} = \dfrac{dP'}{dx}$ gives, on substituting for x' its value, and making $y' = 0$ in the results,

$$Ax + By + C = A_1(x \quad a) \ \ldots\ldots\ldots (2).$$

The equation $\dfrac{dP}{dy} = \dfrac{dP'}{dy}$ gives, moreover,

$$A'y + Bx + C' = A_1(y - b) \ \ldots\ldots \ldots (3).$$

From the equation (2) we have, on differentiating,

$$\frac{dy}{dx} = \frac{A_1 - A}{B} = \tan x\,x';$$

and from (3), by a like process,

$$\frac{dy}{dx} = \frac{B}{A_1 - A'} = \tan x\,x';$$

equating these two expressions, we have

$$\frac{A_1 - A}{B} = \frac{B}{A_1 - A'},$$

which becomes, on reduction,

$$A_1^2 - (A + A')\,A_1 + AA' - B^2 = 0.$$

This equation virtually includes the system given in the former investigations, and its two roots determine A_1 and A_1'.

Had we, indeed, in lieu of the condition last named, made $x' = 0$, which would have given the equation

$$y'^2 = (x - a)^2 + (y - b)^2,$$

we should, as is evident from the symmetrical form of the equations, have obtained

$$\frac{dy}{dx} = \frac{A_1' - A}{B} = \tan xy',$$

$$\frac{dy}{dx} = \frac{B}{A_1' - A'} = \tan xy';$$

whence, by reduction,

$$A_1'^2 - (A + A')A_1' + AA' - B^2 = 0.$$

This equation will give the same values for A_1' as the one last obtained for A_1, and shews that the solution of either is sufficient to determine the values required.

In the equations (1), (2), and (3), assume $x = a$ and $y = b$, which we are allowed to do, since the values of a and b express the position of a point in the axis of x', for which alone these equations are true; then, observing that x' becomes equal to 0, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \text{ ......(4)},$$

$$Aa + Bb + C = 0 \text{ ....(5)},$$

$$A'b + Ba + C' = 0 \text{ ....(6)}.$$

From (5) and (6), by elimination,

$$a = \frac{BC' - A'C}{AA' - B^2}, \quad b = \frac{BC - AC'}{AA' - B^2}.$$

Multiplying (5) by a, and (6) by b, and subtracting half the sum from (4), we have

$$D_1 = Ca + C'b + D.$$

The process of differentiation employed in the solution of this problem, I propose to call *Differentiating along the New Axes.*

Ex. 2. To reduce the general equation of the second order for three variables,

$$Ax^2 + A'y^2 + A''z^2 + 2Bxy + 2B'xz + 2B''yz$$
$$+ 2Cx + 2C'y + 2C''z + D = 0,$$

to the more simple form

$$A_1x'^2 + A_1'y'^2 + A_1''z'^2 + D_1 = 0.$$

Equating these expressions, and differentiating along the axis x', by aid of the condition

$$x'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$$

a, b, and c being co-ordinates of the new centre, we obtain

$$Ax + By + B'z + C = A_1(x - a) \text{ ...... (1)},$$

$$A'y + Bx + B''z + C' = A_1(y - b) \text{ ...... (2)},$$

$$A''z + B'x + B''y + C'' = A_1(z - c) \text{ ...... (3)}.$$

For the new centre at which $x = a$, $y = b$, $z = c$, the above equations become

$$Aa + Bb + B'c + C = 0 \;\ldots\ldots\; (4),$$
$$A'b + Ba + B''c + C' = 0 \;\ldots\ldots\; (5),$$
$$A''c + B'a + B''b + C'' = 0 \;\ldots\ldots\; (6),$$

whence a, b, and c are determined; and by proceeding as in the last example,

$$D_1 = Ca + C'b + C''c + D.$$

If we subtract the equations (4), (5), (6) respectively from the corresponding ones (1), (2), and (3), we have, after transposing to one side,

$$(A - A_1)(x - a) + B(y - b) + B'(z - c) = 0 \ldots\ldots (7),$$
$$(A' - A_1)(y - b) + B(x - a) + B''(z - c) = 0 \ldots\ldots (8),$$
$$(A'' - A_1)(z - c) + B'(x - a) + B''(y - b) = 0 \ldots\ldots (9).$$

Whence, on eliminating $x - a$, $y - b$, $z - c$, we obtain

$$A_1^3 - (A + A' + A'')A_1^2 + (AA' + AA'' + A'A'' - B^2 - B'^2 - B''^2)A_1$$
$$- AA'A'' - 8BB'B'' + AB''^2 + A'B'^2 + A''B^2 = 0 \ldots\ldots (10),$$

a cubic, whose roots determine A_1, A_1', A_1''.

Finally, if from (7) and (8) we eliminate $z - c$, and from (8) and (9) $y - b$, and compare the results, we shall obtain as the symmetrical equations of the axis of x',

$$\{(A - A_1)B'' - BB'\}(x - a) = \{(A' - A_1)B' - BB''\}(y - b)$$
$$= \{(A'' - A_1)B - B'B''\}(z - c).$$

In the last obtained system, it is only necessary to change A_1 into A_1' and A_1'' to exhibit the symmetrical equations of the axes of y' and z'.

In the preceding applications of theory, it has only been necessary to differentiate the even powers, x'^2, y'^2, of the new co-ordinates; an operation which is immediately effected by the aid of the characteristic equation

$$x'^2 + y'^2 + z'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2.$$

When odd powers occur, it is most convenient to employ the first of the linear formulæ of transformation,

$$x' = \cos xx' . (x - a) + \cos yx' . (y - b) + \cos zx' . (z - c);$$

whence

$$\frac{dx'}{dx} = \cos xx', \quad \frac{dx'}{dy} = \cos yx', \quad \frac{dx'}{dz} = \cos zx'.$$

Ex. To reduce the general equation of the second order for three variables to the form

$$A_1x'^2 + A_1'y'^2 + A_1''z'^2 + 2C_1x' = 0.$$

The solution of this problem will differ from that of the last only in consequence of the constant terms, $C_1 \cos xx'$, $C_1 \cos yx'$, $C_1 \cos zx'$, respectively added to the second members of (1), (2), (3). Hence the cubic equation determining A_1, A_1', A_1'', and the

symmetrical equations of the axes x', y', z', will be the same as before. It will at once be seen, that the constants a, b, c, and C_1, will be determined by the system,

$$\begin{aligned}
Aa + Bb + B'c + C &= C_1 \cos xx',\\
A'b + Ba + B''c + C' &= C_1 \cos yx',\\
A''c + B'a + B''b + C'' &= C_1 \cos zx',
\end{aligned}$$

$$Aa^2 + A'b^2 + A''c^2 + 2Bab + 2B'ac + 2B''bc + 2Ca + 2C'b + 2C''c + D = 0.$$

The quantities $\cos xx'$, $\cos yx'$, $\cos zx'$, are known, being determined by the coefficients of the equation of the axis x'.

The last of the above equations reduces to a simple one by the process adopted in the last example.

From the examination of (10) in the preceding example, it is apparent, that when the equation of the second order designates a paraboloid, its coefficients must satisfy the condition

$$AA'A'' + 8BB'B'' - AB''^2 - A'B'^2 - A''B^2 = 0;$$

and that the determination of A_1, A_1', A_1'', one of which, in this instance, becomes 0, will be effected by a quadratic.

In the preceding investigations we have supposed the form of the reduced equation known. In the following example I shall give an illustration of a more general method of solution, by which all the possible forms of the equation are determined, together with the general laws of the coefficients.

Assume

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = A_1x'^2 + A_1'y'^2 + 2B_1x'y' + 2C_1x' + 2C_1'y' + D_1 \ldots (A).$$

Differentiating along the axis of x' with respect to x and y, we have

$$\begin{aligned}
Ax + By + C &= A_1(x-a) + B_1x' \cos xy' + C_1 \cos xx' + C_1' \cos xy',\\
A'y + Bx + C' &= A_1(y-b) + B_1x' \cos yy' + C_1 \cos yx' + C_1' \cos yy'.
\end{aligned}$$

Now $x' \cos xy' = -(y-b)$ and $x' \cos yy' = x - a$;

whence the above equations become

$$\begin{aligned}
Ax + By + C &= A_1(x-a) - B_1(y-b) + C_1 \cos xx' + C_1' \cos xy' \ldots (1),\\
A'y + Bx + C' &= A_1(y-b) + B_1(x-a) + C_1 \cos yx' + C_1' \cos yy' \ldots (2).
\end{aligned}$$

Differentiating (1) and (2), we get

$$\frac{dy}{dx} = \frac{A_1 - A}{B_1 + B} = \frac{B_1 - B}{A' - A_1} = \tan xx' \ldots\ldots (3);$$

whence, on reduction,

$$A_1^2 - (A + A')A_1 + AA' - B^2 + B_1^2 = 0 \ldots\ldots (4).$$

Either of the expressions for $\frac{dy}{dx}$, given in (3), determines the

value of tan xx'. The equation (4) determines A_1 and A_1', and is evidently equivalent to the remarkable system

$$A_1 + A_1' = A + A' \text{ .... } (5),$$
$$A_1 A_1' - B_1^2 = AA' - B^2 \text{ .... } (6).$$

In (A), and in (1) and (2), making $x = a$, $y = b$, and observing that under these suppositions x' vanishes, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \text{......} (7),$$
$$Aa + Bb + C = C_1 \cos xx' + C_1' \cos xy' \text{.........} (8),$$
$$A'b + Ba + C' = C_1 \cos yx' + C_1' \cos yy' \text{.........} (9),$$

of which the first (7) is reducible, as in former examples, to a simple equation.

From the inspection of (3) it appears that the terms A_1, A_1', B_1, and tan xx', are connected by two necessary relations, and that we are therefore at liberty to impose two others. The quantities a, b, C_1, C_1', D_1, are connected by three equations, (7), (8), (9); here, therefore, we may impose likewise two new relations. If we assume

$$B_1 = 0, \quad C_1 = 0, \quad C_1' = 0,$$

we obtain the equations previously given for the discussion of the central system, (Ex. 1). If, on the contrary, we make

$$A_1 = 0, \quad C_1' = 0, \quad D_1 = 0,$$

we obtain a solution true for the case of the parabola.

As yet no use has been made of the higher system of equations

$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \quad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy}, \quad \frac{d^2P}{dy^2} = \frac{d^2P'}{dy^2}, \text{ \&c.}$$

From these a very interesting class of solutions may be obtained. The resulting equations will generally involve a quadratic surd, and will afford a remarkable illustration of the varied and multiform combinations under which the same class of mathematical truths may be presented.

I shall here subjoin a few additional remarks and illustrations with reference to the preceding investigations. The following problem has not, so far as I am aware, been directly solved before.

Given the equations of the projections of a line of the second order on two rectangular co-ordinate planes, to determine the equations of the principal axes; together with the corresponding primitive equation of the curve.

I shall at present merely consider the case in which the equations are of the form

$$ax^2 + a'y^2 + 2bxy + c = 0 \text{ ...... } (1),$$
$$a_1x^2 + a_1'z^2 + 2b_1xz + c_1 = 0 \text{ ...... } (2).$$

The relation among the quantities x, y, z, is evidently linear,

since the curve is supposed to be coincident with a plane. Assume, therefore,

$$z = px + qy,$$

and substituting in (2), we have

$$(a_1 + a_1'p^2 + 2b_1 p)\, x^2 + (2a_1'p + 2b_1)\, qxy + a_1'q^2y^2 + c_1 = 0;$$

hence, by comparison with (1),

$$a = a_1 + a_1'p^2 + 2b_1 p, \quad b = (a_1'p + b_1)\, q, \quad a' = a_1'q^2, \quad c = c_1,$$

the solution of which gives

$$p = b\,\frac{\sqrt{a_1'} - b_1\sqrt{a'}}{a_1' \,.\, \sqrt{a'}}, \qquad q = \sqrt{\frac{a'}{a_1'}},$$

together with the necessary equations of condition,

$$\frac{aa' - b^2}{a'} = \frac{a_1 a_1' - b_1^2}{a_1'}, \quad c = c_1.$$

The fundamental equation

$$x^2 + y^2 + z^2 = x'^2 + y'^2$$

becomes, on substituting as before for z,

$$(1 + p^2)\, x^2 + 2pq\, xy + (1 + q^2)\, y^2 = x'^2 + y'^2 \;\ldots\ldots\; (3).$$

We are now prepared to apply the principle of transformation, which it is the object of this paper to develope.

Assume, therefore,

$$ax^2 + a'y^2 + 2bxy + c = Ax'^2 + A'y'^2 + C \ldots\ldots\ldots (4).$$

Differentiating along the axis of x' with respect to x and y, we obtain

$$\left.\begin{aligned} ax + by &= A\,(1 + p^2)\, x + Apqy \\ bx + a'y &= Apqx + A\,(1 + q^2)\, y \end{aligned}\right\}$$

or

$$\{a - A\,(1 + p^2)\}\, x + (b - Apq)\, y = 0 \ldots\ldots (5),$$
$$(b - Apq)\, x + \{a' - A\,(1 + q^2)\}\, y = 0 \ldots\ldots (6),$$

whence, eliminating x and y,

$$\{a - A\,(1 + p^2)\}\, \{a' - A\,(1 + q^2)\} - (b - Apq)^2 = 0 \;\ldots\ldots\; (7),$$

an equation whose roots determine A and A′.

Of equations (5) and (6), either is sufficient, when combined with the linear equation

$$z = px + qy,$$

to determine the position of the axis x'. Thus, from (6) we have

$$y = \frac{Apq - b}{a' - A\,(1 + q^2)} \cdot x \ldots\ldots (8);$$

and this expression, substituted in the value of z, gives

$$z = \frac{(a' - A)\, p - bq}{a' - A\,(1 + q^2)}$$

whence, by comparison with (8),

$$\frac{x}{a' - A(1 + q^2)} = \frac{y}{Apq - b} = \frac{z}{(a' - A)p - bq},$$

the symmetrical equation of the axis x'. On changing A into A$'$, we obtain the equation of the axis y'.

Finally, on making x' and y' respectively 0 in (3) and (4), we have evidently

$$c = C.$$

The problem is therefore completely resolved.

Since the general equation of the second order may be represented under the symmetrical form

$$a(x - \alpha)^2 + 2b(x - \alpha)(x - \beta) + a'(y - \beta)^2 + c = 0,$$

it is evident that the solution for the more general form, both of the original and the reduced equations, may be easily derived from the above.

It is unnecessary to show that the method which has been here employed in the transformation of functions of two and three variables, is equally applicable to corresponding functions of any number of variables whatever. The transformation of equations of the third and higher orders, is likewise, on the same principle, made to depend on the solution of a *minimum* number of final equations. Neither of these cases being of any importance, I have not thought proper to extend the investigation beyond its present limits. In the case of equations of the third and higher orders, it may, however, be observed, that there would seem to exist more than one system of axes, and more than one relation among the coefficients, of the new and old equations, according to which the transformation may be effected, so as to result in a proposed form, or to satisfy given conditions.

With one or two remarks on the application of the preceding principles to the transformations required in the problem of rotation, and in the undulatory theory of light, I shall bring this communication to a close.

Let U be a function of x, y, z; then, by Maclaurin's theorem,

$$\begin{aligned} U = (U) &+ \left(\frac{dU}{dx}\right)x + \left(\frac{dU}{dy}\right)y + \left(\frac{dU}{dz}\right)z \\ &+ \frac{1}{1.2}\left\{\left(\frac{d^2U}{dx^2}\right)x^2 + \left(\frac{d^2U}{dy^2}\right)y^2 + \left(\frac{d^2U}{dz^2}\right)z^2 + 2\left(\frac{d^2U}{dx\,dy}\right)xy \right. \\ &\qquad \left. + 2\left(\frac{d^2U}{dx\,dz}\right)xz + 2\left(\frac{d^2U}{dy\,dz}\right)yz\right\} \\ &+ \text{\&c. \&c.} \end{aligned}$$

Imagine U to be transformed into a function of x', y', z', the equations of transformation being homogeneous and of the first degree, then

$$U = (U) + \left(\frac{dU}{dx'}\right) x' + \left(\frac{dU}{dy'}\right) y' + \left(\frac{dU}{dz'}\right) z'$$
$$+ \frac{1}{1.2}\left\{\left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2 + 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + \&c.\right\}$$

Now, from the nature of the relation between x, y, z and x', y', z', the above values of U cannot be equal, unless each aggregate of homogeneous terms in the one be equivalent to the corresponding aggregate in the other. Those of the second order give, on comparison,

$$\left(\frac{d^2U}{dx^2}\right) x^2 + \left(\frac{d^2U}{dy^2}\right) y^2 + \left(\frac{d^2U}{dz^2}\right) z^2$$
$$+ 2\left(\frac{d^2U}{dx\,dy}\right) xy + 2\left(\frac{d^2U}{dx\,dy}\right) xz + 2\left(\frac{d^2U}{dy\,dz}\right) yz$$
$$= \left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2$$
$$+ 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + 2\left(\frac{d^2U}{dx'\,dz'}\right) x'z' + 2\left(\frac{d^2U}{dy'\,dz'}\right) y'z'.$$

In the transformation of an equation of the second degree, it is therefore necessary that the coefficients should express the values assumed by a certain system of partial differential coefficients, when the variables vanish; and conversely, when such special values of the required system of partial differential coefficients present themselves, they may be transformed by considering them as coefficients of the equation of the second degree. The former of these cases is the one met with in the problem of rotation, the latter in the undulatory theory of light.

IV.—ON THE FAILURE OF FORMULÆ IN THE INVERSE PROCESSES OF THE DIFFERENTIAL CALCULUS.

If we apply the rule for integrating any power of x to the particular case when the index of the power is -1, we obtain a result having 0 in the denominator, and which is therefore nugatory. This is only one instance of several in which a certain relation of the subject to an inverse operation makes the general formulæ fail; and as these cases give rise to some difficulty, we shall here consider two of the most important of them. The instance to which we have alluded is so well known, that we need do no more than mention it; and for the more general case of failure when the index is of any value, the reader is referred to Art. VI. Vol. I. p. 109. The method of arriving at the true value in these cases of failure, is the same as that which we shall pursue in those

are about to consider. The principle is this: since the function in this particular case becomes infinite, we may so assume the arbitrary constant in the complementary function, as to make the formula for the particular value take the indeterminate form $\frac{0}{0}$, the true value of which can easily be determined by the ordinary rules. The assumption made with respect to the arbitrary constant in the complementary function, is to make it negative and infinite, so that the difference of two infinite quantities may be finite. Exactly the same principle holds in the instances we are about to consider.

Suppose we had the differential equation

$$\frac{dy}{dx} - ay = \varepsilon^{ax},$$

we should find, by the usual rule for integrating such equations,

$$y = \frac{\varepsilon^{ax}}{a - a} + C\varepsilon^{ax},$$

the form of which is nugatory. To discover the true value, let us suppose that the multiplier of y is not the same as the multiplier of x, but that the equation is

$$\frac{dy}{dx} - a_1 y = \varepsilon^{ax};$$

the integral of which is

$$y = \frac{\varepsilon^{ax}}{a - a_1} + C\varepsilon^{a_1 x}.$$

Now C being arbitrary, we may conceive it to consist of two parts, so that $C = -\frac{1}{a - a_1} + C_1$, which gives

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{a - a_1} + C_1\varepsilon^{a_1 x}.$$

Now when $a_1 = a$, the first term takes the form $\frac{0}{0}$, which is indeterminate; and by the usual method its true value, when $a_1 = a$, is found to be $x\varepsilon^{ax}$, so that

$$y = x\varepsilon^{ax} + C_1\varepsilon^{ax},$$

which is the true solution of the equation.

If the operating factor were of the order r, so that the equation was

$$\left(\frac{d}{dx} - a\right)^r y = \varepsilon^{ax},$$

we should find by the usual rule

$$y = \frac{\varepsilon^{ax}}{(a - a)^r} + (C_0 + C_1 x + \&c. + C_{r-1} x^{r-1})\,\varepsilon^{ax},$$

a nugatory result.

If we suppose the a in the operating factor to be different from the multiplier of x, we should have, by a change of the first arbitrary constant,

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{(a - a_1)^r} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x}.$$

If we differentiate the numerator and denominator of the first term in order to determine its value when $a_1 = a$, we find

$$y = \frac{x \varepsilon^{ax}}{r(a - a_1)^{r-1}} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

which is still nugatory when $a = a_1$. We must therefore continue the process, changing the constant in the second term of the complementary function, when we obtain

$$y = \frac{x(\varepsilon^{ax} - \varepsilon^{a_1 x})}{r(a - a_1)^{r-1}} + (C_0' + C_1' x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

the first term of which we find, as before, to be infinite when we make $a = a_1$. But by continuing the same process as before, we shall at last obtain

$$y = \frac{x^r \varepsilon^{ax}}{r(r-1)\ldots 2.1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

which, when $a_1 = a$, becomes

$$y = \frac{x^r \varepsilon^{ax}}{r(r-1)\ldots 2.1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1}) \varepsilon^{ax},$$

being the true solution of the equation.

The other example which we shall here consider is particularly important, as the form of the solution occurs in the second approximation in the Lunar Theory, rendering necessary a change in the form of the equation.

It is met with in the integration of the equation

$$\frac{d^2 u}{d\theta^2} + n^2 u = \cos m\theta,$$

when $m = n$. For the general solution is

$$u = \frac{\cos m\theta}{n^2 - m^2} + C \cos n\theta + C_1 \sin n\theta,$$

the first term of which is infinite when $m = n$. But if, as before, we change the arbitrary constants in the complementary function, we can put the equation under the form

$$u = \frac{\cos m\theta - \cos n\theta}{n^2 - m^2} + C' \cos n\theta + C_1 \sin n\theta.$$

The value of the first term of this, when $m = n$, determined in the usual way, is $\dfrac{\theta \sin n\theta}{2n}$; so that

$$u = \frac{\theta \sin n\theta}{2n} + C' \cos n\theta + C_1 \sin n\theta$$

In the same way, if the original equation were

$$\frac{d^2u}{d\theta^2} + n^2u = \sin n\theta,$$

we should find

$$u = -\frac{\theta \cos n\theta}{2n} + C \cos n\theta + C_1' \sin n\theta.$$

If the original equation were

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^2 u = \cos m\theta,$$

we should have

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \left(\frac{d^2}{d\theta^2} + n^2\right)^{-1} (C_0 \cos n\theta + C_1 \sin n\theta);$$

which, by what we have just found, (observing that the constants are arbitrary,) is equal to

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \theta\,(C_0 \sin n\theta - C_0' \cos n\theta) + C_1' \sin n\theta + C_1' \cos n\theta.$$

The true value of the first term of this, when $m = n$, will be found, by the same process as in the last example, to be

$$-\frac{\theta^2 \cos n\theta}{2 \,.\, 1\, (2n)^2};$$

and generally, if the equation be

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^r u = \cos m\theta,$$

the true value of the first term will be, when $m = n$,

$$\frac{\theta^r \cos\left(n\theta + r\dfrac{\pi}{2}\right)}{r\,(r-1) \ldots\ldots 2 \,.\, 1\, (2n)^r}.$$

D. F. G.

V.—ON DIAMETRAL CURVES.

Let the equation to any curve of the n^{th} order be represented by

$$f(x, y) = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Let the equation to any straight line cutting any number of the branches of the curve be

$$y = mx + p \;\ldots\ldots\ldots\ldots\; (2).$$

Let α, β be the co-ordinates of the middle point of any portion of this straight line, intercepted between any two of the branches of the curve.

Transfer the origin of co-ordinates to the point (α, β). Then the equation (1) will become

$$f(\alpha + x', \beta + y') = 0 \text{ .... } (3),$$

and the equation (2) will become

$$\beta + y' = m(\alpha + x') + p;$$

or, since evidently

$$\beta = m\alpha + p,$$

it will become

$$y' = mx' \text{ .... } (4).$$

Combining (3) and (4), we have the following equation in x' for the intersection of the straight line and the curve,

$$f(\alpha + x', \beta + mx') = 0,$$

$$\text{or } f(\alpha, \beta) + x'\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)f(\alpha, \beta) + \frac{x'^2}{1.2}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^2 f(\alpha, \beta)$$

$$+ \frac{x'^3}{1.2.3}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^3 f(\alpha, \beta) + \text{\&c.}$$

$$+ \frac{x'^n}{1.2.3 \ldots n}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^n f(\alpha, \beta) = 0.$$

Hence, putting

$$f(\alpha, \beta) = k_0,$$

$$\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)f(\alpha, \beta) = k_1,$$

$$\frac{1}{1.2}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^2 f(\alpha, \beta) = k_2,$$

$$\text{\&c.} = \text{\&c.}$$

$$\frac{1}{1.2.3 \ldots n}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^n f(\alpha, \beta) = k_n,$$

our equation becomes

$$k_0 + k_1 x' + k_2 x'^2 + k_3 x'^3 + \ldots + k_n x'^n = 0.$$

Now, from the peculiar position of the new origin of co-ordinates, it is clear that this equation must contain two equal roots with opposite signs, and therefore we may represent the roots of the equation by

$$\rho_1, \rho_2, \rho_3 \ldots\ldots \rho_{n-2}, a, -a.$$

Let S_1 = sum of $-\rho_1, -\rho_2, -\rho_3 \ldots -\rho_{n-2}, -a, a,$

S_2 = sum of the products of these quantities taken two together,

S_3 = ditto taken three together,

&c. = &c.

S_n = ditto taken n together.

Then, by the theory of equations, we have the following relations:

$$k_0 = S_n k_n,$$
$$k_1 = S_{n-1} k_n,$$
$$k_2 = S_{n-2} k_n,$$
$$\&c. = \&c.$$
$$k_{n-1} = S_1 . k_n;$$

and therefore, if between these n equations we eliminate the $n - 1$ quantities $\rho_1, \rho_2, \rho_3, \ldots\ldots \rho_{n-2}, a$, we shall get an equation

$$\phi(a, \beta) = 0 \ldots\ldots\ldots\ldots (5),$$

which will be the equation to the diametral curve.

It is evident that the order of the equation (5) will be designated by the formula $\frac{n(n-1)}{1.2}$, which expresses the number of different pairs of intersections which the straight line (2) experiences from the different branches of the curve (1).

Ex. To determine the equation to the diametral curve of the curve

$$y^3 = bx,$$

we have

$$S_n = (-\rho_1).(-a).(a) = a^2\rho_1,$$
$$S_{n-1} = (-a).(-\rho_1) + a(-\rho_1) + (-a).(a) = -a^2,$$

and

$$S_{n-2} = a - a - \rho_1 = -\rho_1;$$
$$\therefore S_n = S_{n-1} . S_{n-2};$$
$$\therefore k_0 . k_n = k_1 . k_2,$$

or, since $n = 3$,

$$k_0 . k_3 = k_1 . k_2.$$

But $k_0 = f(a, \beta) = \beta^3 - ba,$

$$k_1 = \left(\frac{d}{da} + m\frac{d}{d\beta}\right) f(a, \beta) = 3m\beta^2 - b,$$

$$k_2 = \frac{1}{1.2}\left(\frac{d}{da} + m\frac{d}{d\beta}\right)^2 f(a, \beta) = 3m^2\beta,$$

and $k_3 = \frac{1}{1.2.3}\left(\frac{d}{da} + m\frac{d}{d\beta}\right)^3 f(a, \beta) = m^3.$

Hence, the required equation is

$$(\beta^3 - ba)\, m^3 = (3m\beta^2 - b) . 3m^2\beta,$$
$$\therefore m(\beta^3 - ba) = 3\beta(3m\beta^2 - b);$$
$$\therefore 8m\beta^3 - 3b\beta + mba = 0.$$

W. W.

VI.—ON A CERTAIN PROPERTY IN THE THEORY OF NUMBERS.*

MR. Barlow, in his *Theory of Numbers*, has treated of the properties of the equation

$$x^m - b = \text{M} \,.\, (n),$$

the letter M being used as an abbreviation of the words "multiple of." This is perhaps more familiar to many persons under the form

$$\frac{x^m - b}{n} = \text{an integer.}$$

In what follows it is proposed to extend some of the properties of this equation to the more general one,

$$\text{A}_0 x^m + \text{A}_1 x^{m-1} + \text{A}_2 x^{m-2} + \ldots + \text{A}_{m-1} x + \text{A}_m = \text{M} \,.\, (n),$$

n being a prime number, and the quantities A_0, A_1, $\text{A}_2 \ldots \text{A}_m$ being integers, of which A_0 is supposed prime to n. The integer values of x *less than* n, which satisfy this equation, I call *primary* roots; those which are $> n$ but $< 2n$, *secondary* roots, and so on.

1. The equation

$$\text{A}_0 x^m + \text{A}_1 x^{m-1} + \ldots + \text{A}_m = \text{M} \,.\, (n),$$

cannot have more than m primary roots. For, if possible, let it have $m + 1$ primary roots, viz. r_1, r_2, $r_3 \ldots r_{m+1}$;

$$\therefore\ \text{A}_0 r_1^m + \text{A}_1 r_1^{m-1} + \ldots + \text{A}_{m-1} r_1 + \text{A}_m = \text{M} \,.\, (n),$$

$$\text{and } \text{A}_0 r_2^m + \text{A}_1 r_2^{m-1} + \ldots + \text{A}_{m-1} r_2 + \text{A}_m = \text{M} \,.\, (n).$$

Subtracting the latter of these from the former, we have

$$\text{A}_0 (r_1^m - r_2^m) + \text{A}_1 (r_1^{m-1} - r_2^{m-2}) + \ldots + \text{A}_{m-1} (r_1 - r_2) = \text{M}\ (n).$$

Now, $r_1 - r_2$ is $< n$, and therefore prime to n; we may consequently divide by it.

Hence,

$$\text{A}_0 (r_1^{m-1} + r_2 r_1^{m-2} + \ldots + r_2^{m-1}) \\ + \ldots + \text{A}_{m-2} (r_1 + r_2) + \text{A}_{m-1} = \text{M} \,.\, (n).$$

Had we used r_3 instead of r_2, we should have had

$$\text{A}_0 (r_1^{m-1} + r_3 r_1^{m-2} + \ldots + r_3^{m-1}) \\ + \ldots + \text{A}_{m-2} (r_1 + r_3) + \text{A}_{m-1} = \text{M} \,.\, (n).$$

Subtracting this from the preceding, and dividing by $r_2 - r_3$, we find

$$\text{A}_0 \{ r_1^{m-2} + (r_2 + r_3) \,.\, r^{m-3} + \ldots + r_2^{m-2} + \ldots + r_3^{m-2} \} \\ + \ldots + \text{A}_{m-2} = \text{M} \,.\, (n).$$

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* From a Correspondent.

Writing r_4 for r_3, subtracting and dividing as before, we shall reduce the equation to the dimensions $m - 3$ in r, and lose the coefficient A_{m-2} from the end. It is sufficiently obvious, that by pursuing this process as far as to the substitution of r_m, and then dividing by $r_{m-1} - r_m$, we shall have lost successively the coefficients A_m, A_{m-1}, ... A_2, and the equation will be of one dimension in r_1; and as the term, of which A_0 is the coefficient, involves symmetrically all the quantities substituted, it must therefore be the following,

$$A_0(r_1 + r_2 + r_3 + \dots + r_m) + A_1 = M.(n).$$

Had we employed r_{m+1} *instead* of r_m, we should have found

$$A_0(r_1 + r_2 + \dots + r_{m-1} + r_{m+1}) + A_1 = M.(n);$$

$$\therefore\ A_0(r_m - r_{m+1}) = M.(n).$$

But this is absurd, for both A_0 and $(r_m - r_{m+1})$ are prime to n, by hypothesis; and consequently the proposed equation cannot have so many as $m + 1$ primary roots.

Cor. If we have an equation

$$A_0x^m + A_1x^{m-1} + \dots + A_{m-1}x + A_m = M.(n),$$

which we know to have $m + 1$ primary roots, A_0 must be divisible by n, since, as before,

$$A_0(r_m - r_{m+1}) = M.(n).$$

But in this case the equation reduces itself to

$$A_1x^{m-1} + A_2x^{m-2} + \dots + A_{m-1}x + A_m = M.(n);$$

for the first term, A_0x^m, being always a multiple of n, may be rejected. This new equation having the $m + 1$ primary roots of the original one, we must have A_1 a multiple of n; and, by continuing this reasoning, it will follow that all the coefficients A_0, A_1, A_2, ... A_m, are multiples of n.

2. When the equation

$$A_0x^m + A_1x^{m-1} + \dots + A_{m-1}x + A_m = M.(n),$$

has *exactly* m primary roots (A_0 being prime to n), and S_r be used to denote the sum of the products of those roots taken r and r together, then the quantities

$$A_0S_1 + A_1,\quad A_0S_2 - A_2,\quad A_0S_3 - A_3, \dots A_0S_m + (-1)^{m+1}A_m,$$

are all multiples of n.

Let x denote any one of the primary roots r_1, r_2, r_3, ... r_m; then it is evident that

$$0 = A_0(x - r_1)(x - r_2)\dots(x - r_m),$$

$$= A_0x^m - A_0S_1x^{m-1} + A_0S_2x^{m-2} - \dots \pm A_0S_m.$$

But $M.(n) = A_0x^m + A_1x^{m-1} + A_2x^{m-2} + \dots + A_m$.

Hence, taking the difference of these two equations, we find

$$M.(n) = (A_0S_1 + A_1)x^{m-1} - (A_0S_2 - A_2)x^{m-2} + \dots \mp (A_0S_m \mp A_m).$$

Now x denotes *any* one of the quantities r_1, r_2, ... r_m, and consequently this equation, which is of $m-1$ dimensions, has m primary roots; therefore, by the last corollary, its coefficients are multiples of n.

3. It seldom happens that an equation of the form we have proposed has as many primary roots as dimensions: there is, however, one which fulfils this condition, and is familiar to most readers,—it is

$$x^{n-1} - 1 = \mathrm{M}.(n),$$

which constitutes Fermat's theorem.

In this particular equation $A_0=1$, $A_1=A_2=A_3=...=A_{m-1}=0$, and $A_m = -1$: and the primary roots are 1, 2, 3, ... $(n-1)$. Hence

$$S_1 = 1 + 2 + 3 + ... + (n-1),$$
$$S_2 = 1.2 + 1.3 + 2.3 + ... + (n-2)(n-1),$$
$$..............................$$
$$S_{n-2} = 1.2.3 ... (n-1)\left(1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{n-1}\right),$$
$$S_{n-1} = 1.2.3 ... (n-1).$$

And, by the general properties proved above, the quantities S_1, S_2, S_3, ... S_{n-2}, are all multiples of n: also, since $A_m=-1$, we have corresponding to the term $A_0S_m - A_m$, the quantity

$$\{1.2.3 ... (n-1)\} + 1,$$

a multiple of n, which is Wilson's theorem.

This is a complete solution of one of the questions proposed in the *Cambridge Problems* for 1836.

V.

VII.—ON FRESNEL'S METHOD OF FINDING, BY APPROXIMATION, THE VALUES OF THE DEFINITE INTEGRALS,

$$\int_a^b dx \,.\, {\sin \atop \cos} \{\pi . f(x)\},$$

WHERE $f(x)$ IS ANY FUNCTION OF x EXPRESSIBLE IN A SERIES OF A FINITE NUMBER OF TERMS.*

THE method employed by Fresnel for obtaining the two integrals,†

$$\int dx \,.\, \sin\left(\frac{\pi}{2}x^2\right) \text{ and } \int dx \,.\, \cos\left(\frac{\pi}{2}x^2\right),$$

* From a Correspondent.

† See Memoires de L'Institut for 1821 and 1822, p. 407.

between any given limits, is so simple and apparently so obvious, that it would seem scarcely necessary to discuss it here, if we had not known a recent instance of an integral of this form puzzling skilful analysts for several years, and which at length was only computed by an extraordinary expenditure of labour.

Fresnel lays no claim to novelty in the method, but until it is shewn that it had been discovered by some one else previously, we must attribute the full merit of it to his fertile genius. The fact of its being overlooked shews how slightly Fresnel's last paper on diffraction has been studied by those who felt justified in speaking positively on its contents, and with whom an integration between the limits, x equal to a small quantity and x equal to infinity, was accepted as an adequate approximation to an integration, which the experiments in reality required, between the limits x equal to a small quantity and x equal to about 180. To have given the arguments built upon this case the slightest legitimate weight, this substitution ought to have been discussed.

The problem under discussion was the diffraction by a single straight edge of an opaque plate; and the experiments in fact were made with an aperture with parallel straight edges, having only the distance of a centimetre, or about 4-10ths of an inch. The second edge of such an aperture does not, doubtless, sensibly affect the diffraction by the other edge at such distances as were used: but when the arguments are drawn from very small differences, as in this case, it is clear that it should be shewn from theory as well as from experiments, that such an aperture might be substituted for a single edge. Fresnel says, p. 429, "J' avois soin que les plaques fussent séparées par un intervalle assez grand pour que l' une n' eût aucune influence sur les franges produites par l' autre. Dans presque toutes mes observations, cet intervalle était d' un centimètre."

First, let the integral $\int dx \,.\cos \{\pi \,. f(x)\}$ be taken in its most general form,

$$\int dx \,.\cos \{\pi (a + bx^{\alpha} + cx^{\beta} + dx^{\gamma} + \&c.)\},$$

where α, β, γ are integral or fractional, positive or negative.

Let now $x = m + v$, where m is any constant, and v a variable quantity, which is always very small: then

$$\begin{aligned}
&\int dx \,.\cos \{\pi \,. f(x)\} \\
&\quad = \int dv \,.\cos \{\pi \left[a + b\,(m+v)^{\alpha} + c\,.(m+v)^{\beta} + d\,.(m+v)^{\gamma} + \&c.\right]\} \\
&\quad = \int dv \,.\cos \left[\pi \{a + bm^{\alpha} + cm^{\beta} + dm^{\gamma} + \&c.\right. \\
&\qquad\qquad + v\,.\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1}) \\
&\qquad\qquad \left. + \text{ terms containing } v^2,\ v^3,\ \&c.\}\right]
\end{aligned}$$

Since v is very small, the terms involving v^2, v^3, &c. will not sensibly increase or diminish the sum of the constant term plus that involving the first power of v;

$$\therefore \int dx \,.\, \cos\{\pi \,.\, f(x)\}$$
$$= \int dv \,.\, \cos\{\pi\,[a + bm^{\alpha} + cm^{\beta} + dm^{\gamma} + \&c. + v\,.\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1} + \&c.)]\}$$
$$= \frac{1}{\pi\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1} + \&c.)}$$
$$\times \sin\{\pi\,[a + bm^{\alpha} + cm^{\beta} + dm^{\gamma} + \&c. + v\,.\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1} + \&c.)]\} + C.$$

Let the integral be taken between the limits $x = m$ and $x = m + h$, or between $v = 0$ and $v = h$, then

$$\int_m^{m+h} dx \,.\, \cos\{\pi \,.\, f(x)\} = \frac{1}{\pi\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)}$$
$$\times \left(\begin{array}{r}\sin\{\pi\,[a + bm^{\alpha} + cm^{\beta} + \&c. \ldots + h\,.\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)]\} \\ -\sin\{\pi(a + bm^{\alpha} + cm^{\beta} + \&c.)\}\end{array}\right)$$
$$= \frac{2}{\pi\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)}$$
$$\times \cos\{\pi\,[a + bm^{\alpha} + cm^{\beta} + \&c. \ldots + \frac{h}{2}(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)]\}$$
$$\times \sin\left\{\frac{\pi h}{2}(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1} + \&c.)\right\}$$

In the same way we have

$$\int dx \,.\, \sin\{\pi \,.\, f(x)\} = C - \frac{1}{\pi\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)}$$
$$\times \cos\{\pi\,[a + bm^{\alpha} + cm^{\beta} + \&c. + v\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)]\};$$

and between the limits $x = m$ and $x = m + h$,

$$\int_m^{m+h} dx \,.\, \sin\{\pi \,.\, f(x)\} = \frac{2}{\pi\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)}$$
$$\times \sin\{\pi\,[a + bm^{\alpha} + cm^{\beta} + \&c. + \frac{h}{2}(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)]\}$$
$$\times \sin\left\{\frac{\pi h}{2}(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + \&c.)\right\}$$

To apply these formulæ to particular cases, we shall first take Fresnel's integrals,

$$\int dx \,.\, \cos\left(\frac{\pi}{2}x^2\right) \text{ and } \int dx \,.\, \sin\left(\frac{\pi}{2}x^2\right),$$

and we have
$$\begin{aligned} a &= 0, \\ b &= \tfrac{1}{2} \ldots \alpha = 2, \\ c &= 0, \\ d &= 0, \\ &\&c. \end{aligned}$$

$$\therefore \int^{m+h} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right) = \int^{m} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right)$$
$$+ \frac{2}{\pi m}\cos\left\{\frac{\pi m}{2}(m+h)\right\}\sin\left(\frac{\pi h}{2}m\right),$$

$$\int^{m+h} dx \,.\, \sin\left(\frac{\pi}{2}x^2\right) = \int^{m} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right)$$
$$+ \frac{2}{\pi m}\sin\left\{\frac{\pi}{2}m(m+h)\right\}\sin\left(\frac{\pi h}{2}m\right). .$$

Bernoulli's series is convergent for these integrals when m is small, but becomes divergent when m is large; so that when we have calculated the values of the integrals for small values of x by Bernoulli's series, we can continue the computations for larger and larger values without limit, by the above simple formulæ.

For another example, we shall take the Astronomer Royal's integral in his Paper on the Intensity of Light near a Caustic, in the *Cambridge Phil. Trans.* for 1838, namely,

$$\int dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\}.$$

Comparing with our general solution, we have

$$a = 0,$$
$$b = \frac{1}{2}, \qquad \alpha = 3,$$
$$c = -\frac{a}{2}, \quad \beta = 1,$$

and

$$\int^{m+h} dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\} = \int^{m} dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\}$$
$$+ \frac{4}{\pi(3m^2 - a)}\cos\left\{\frac{\pi}{2}\left(m^3 - am + \frac{h}{2}(3m^2 - a)\right)\right\}$$
$$\times \sin\left\{\frac{\pi h}{4}(3m^2 - a)\right\}.$$

In the same manner as with the former integrals, this formula would be used when the value of m became so large as to cause Bernoulli's series to be divergent.

R.

VIII.—ON THE GENERAL THEORY OF THE LOCI OF CURVILINEAR INTERSECTION.

1. The algebraical equation which comprehends the locus of the intersections of two series of curves, connected together by some assigned law, results in all cases from the elimination of an arbitrary quantity a between two equations,

$$f(x, y, a) = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and } \phi(x, y, a) = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

In the original conception of the problem, the symbol a designates some specific quantity, and the equations (1) and (2) represent two specific curves. But in the equation which results from the elimination of a between them, it is evident that all traces are lost of the peculiar meaning which may originally have been attached to it, and that consequently the final equation in x and y must comprehend and represent, not only the peculiar locus for which the investigation may have been particularly instituted, but likewise any other which any variation of the conceived affections of a may render appropriate. It is also equally manifest, that whatever curve the final equation may comprehend, there must always exist geometrical meanings for the equations (1) and (2), from which it has arisen. The truth of this converse proposition will be immediately acknowledged when we recollect, that according to the general theory of algebraical affections every geometrical equation in x and y is significant.

2. We will proceed to apply the above principles to the consideration of a class of loci which belong to the intersection of tangents at any points in the general branches of a curve with straight lines, connected with these tangents by some assigned law, and passing through some assigned point.

Let the equation to the curve, cleared of radicals, be

$$f(x, y) = 0 \ldots\ldots\ldots\ldots (3),$$

and to the tangent line at any point

$$y = ax + \beta \ldots\ldots\ldots\ldots (4).$$

At the intersection of the curves (3) and (4), we have

$$f(x, ax + \beta) = 0 \ldots\ldots(5);$$

and since from the nature of the contact this equation must have two equal roots, it is clear that the elimination of x between the two following equations

$$f(x, ax + \beta) = 0,$$

$$\text{and } \frac{d}{dx} f(x, ax + \beta) = 0,$$

must give rise to an equation

$$\phi(a, \beta) = 0 \ldots\ldots\ldots\ldots (6),$$

which expresses all the appropriate relations between a and β, the dimensions of this equation in β corresponding to the number of parallel curvilinear elements which exist at different points of the curve (3).

Suppose, now, that from (6) we determine a value $\theta(a)$ for β, where $\theta(a)$ denotes some function of a, and we shall have for the equation to a tangent line

$$y = ax + \theta(a)\ldots\ldots\ldots(7).$$

Again, let the equation to a line passing through a point a, b, and connected by some law with the line (7), be represented by

$$y - b = \chi(a)(x - a)\ldots\ldots(8).$$

Then manifestly, if between (7) and (8) we eliminate a, we shall obtain an equation in x and y comprehending all the loci belonging to the infinite number of values of a, as expressed under the general form $+^r\mu$, where r is any number whatever, and μ a symbol of quantity.

It is easy to see, on a little reflection, that if we eliminate β any how between the three equations (4), (6), and (8), instead of pursuing the process above described, we shall obtain a final equation in x and y which will comprehend all the loci we have just been considering, which belong to every solution of the equation (6) for β in terms of a.

3. Suppose that, to take a particular form for the symbol of functionality χ, we assume

$$\chi(a) = -\frac{1}{a};$$

then the equation (8) will become

$$y - b = -\frac{1}{a}(x - a)\ \ldots\ldots\ldots\ (9),$$

and therefore from (7) we have for the locus of the intersection for the value $\theta(a)$ of β, the equation

$$y = -\frac{x(x-a)}{y-b} + \theta\left(-\frac{x-a}{y-b}\right)\ldots\ldots(10).$$

4. If we take $a = (-)^{\frac{1}{2}}\mu$, then the equation (9) becomes

$$y - b = -\frac{1}{(-)^{\frac{1}{2}}\mu}(x - a),$$

$$\text{or } y - b = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - a);$$

and consequently, since for all values of a we must get the same equation (10), it appears that the same final equation must arise for the expression of the loci of the intersections of the curves

$$y - b = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - a),$$

$$\text{and } y = (-)^{\frac{1}{2}}\mu x + \theta\{(-)^{\frac{1}{2}}\mu\};$$

as for that of the loci of the intersections of

$$y - b = -\frac{1}{\mu}(x - a),$$

$$\text{and } y = \mu x + \theta(\mu).$$

5. The general conclusions of the preceding article are susceptible of elegant illustrations in the conic sections.

First, let us take the parabola

$$y^2 = 4mx \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Then the equation to a tangent to it in the plane $(+, +)$ will be

$$y = \mu x + \frac{m}{\mu} \;\ldots\ldots\ldots\; (2),$$

and to a perpendicular upon it through the focus

$$y = -\frac{1}{\mu}(x - m) \;\ldots\ldots\; (3).$$

Now, from Art. 4, we know that the equation resulting from the elimination of μ between these two equations, will be the same as between the equations

$$y = (-)^{\frac{1}{2}}\mu x + \frac{m}{(-)^{\frac{1}{2}}\mu} \;\ldots\ldots\ldots\; (4),$$

$$\text{and } y = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - m) \;\ldots\ldots\ldots\; (5).$$

But the equation (4) is that of a tangent to the parabola (1) in the plane $(+, +^{\frac{1}{4}})$, as will easily be ascertained on determining the value of β, from the consideration that the equation

$$\{(-)^{\frac{1}{2}}\mu x + \beta\}^2 = 4mx$$

has two equal roots.

And the equation (5) represents a straight line in the same plane with (4), which makes angles with the axis of x *complementary* to those made with it by (4).

Hence the algebraical expression of the solution of the problem, "To find the equation to the locus of the intersections of tangents to a parabola in the plane $(+, +)$, with perpendiculars upon them from the focus," will comprehend also the solution of the problem, "To find the equation to the locus of the intersections of tangents to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, with straight lines passing through the focus of the branch in the plane $(+, +)$, and making with the axis of x angles *complementary* to those which are made with it by the tangents."

We will accordingly proceed to eliminate μ between the equations (2) and (3), and subtracting (3) from (2), we have

$$0 = \left(\mu + \frac{1}{\mu}\right) x,$$

$$\text{and therefore } x\left(\frac{x - m}{y} + \frac{y}{x - m}\right) = 0.$$

The solution

$$x = 0$$

belongs to the former problem, and represents the axis of y, while the solution

$$\frac{x - m}{y} + \frac{y}{x - m} = 0,$$

$$\text{or } \ y^2 + (x - m)^2 = 0,$$

belongs to the latter problem, and designates two straight lines in the plane $(+, +^{\frac{1}{4}})$, passing through the focus of the branch in the plane $(+, +)$, and inclined on each side at angles of 45° to the axis of x.

That they *should* be straight lines, is evident from the solution of the following problem in the plane $(+, +)$: "To find the equation to the locus of the intersections of tangents to a parabola with straight lines passing through the foot of the directrix, and making angles with the axis of x complementary to those made by the respective tangents."

The equation to the locus will evidently result from the elimination of μ between the two following equations:

$$y = \mu x + \frac{m}{\mu},$$

$$\text{and } y = \frac{1}{\mu}(x + m).$$

By subtraction we have

$$0 = \left(\mu - \frac{1}{\mu}\right) x,$$

$$\text{and therefore } 0 = \left(\frac{x + m}{y} - \frac{y}{x + m}\right) x,$$

$$\text{and therefore } y^2 = (x + m)^2,$$

which represent two straight lines in the plane $+, +$. The solution

$$x = 0,$$

on the contrary, denotes the locus of the intersections of perpendiculars from the focus of the parabolic branch in the plane $(+, +^{\frac{1}{4}})$, with tangents to this branch, being the axis of y in the plane $(+, +^{\frac{1}{4}})$.

We will next take the ellipse, whose equation is

$$a^2y^2 + b^2x^2 = a^2b^2 \dots\dots\dots (1).$$

The equations in this case corresponding to the equations (2) and (3) in the preceding investigations for the parabola, will be respectively

$$y = \mu x + \sqrt{a^2\mu^2 + b^2} \dots\dots\dots\dots\dots (2),$$

$$\text{and } y = -\frac{1}{\mu}(x - \sqrt{a^2 - b^2}) \dots\dots\dots\dots (3),$$

and the equations corresponding to (4) and (5) will be respectively

$$y = (-)^{\frac{1}{2}}\mu x + \sqrt{a^2\{(-)^{\frac{1}{2}}\mu\}^2 + b^2} \dots\dots (4),$$

$$\text{and } y = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - \sqrt{a^2 - b^2}) \dots\dots\dots\dots\dots\dots (5).$$

And, since by Art. 4, the result of the elimination of μ between the former pair of equations (2), (3), and the latter (4), (5), will be the same, we see clearly, that in this case, as in the parabola, the final equation will represent two loci resulting from the solution of two different problems, exactly analogous to the two problems there discussed.

We will proceed to obtain the final equation.

From (2) we have

$$y - \mu x = (a^2\mu^2 + b^2)^{\frac{1}{2}},$$

and from (3),

$$\mu y + x = (a^2 - b^2)^{\frac{1}{2}}.$$

Squaring these two equations and adding the results, we get

$$(1 + \mu^2)(x^2 + y^2) = (1 + \mu^2)a^2,$$

and therefore

$$(x^2 + y^2 - a^2)(1 + \mu^2) = 0;$$

from which, by the aid of (3), we get

$$(x^2 + y^2 - a^2)\left\{1 + \frac{\{(a^2 - b^2)^{\frac{1}{2}} - x\}^2}{y^2}\right\} = 0,$$

and therefore

$$(x^2 + y^2 - a^2)\{y^2 + [(a^2 - b^2)^{\frac{1}{2}} - x]^2\} = 0.$$

The solution

$$x^2 + y^2 - a^2 = 0$$

belongs to the former problem, and the solution

$$y^2 + \{(a^2 - b^2)^{\frac{1}{2}} - x\}^2 = 0$$

to the latter, being the equations to two straight lines in the plane $+, +^{\frac{1}{4}}$, passing through the focus of the ellipse in the plane

N

$(+, +)$, and inclined to the axis of x at angles of 45°, exactly according to the analogy of the lines previously discussed in the case of the parabola.

6. It may not be altogether useless to discuss the algebraical solution of the following problem, for the further elucidation of the principles developed in the preceding disquisitions:

"To determine the locus of the intersections of perpendiculars from the focus of the parabola upon their corresponding tangents, with tangents symmetrically situated on the opposite side of the parabola."

It is evident, on a little reflection, that the equation to the tangent will be

$$y = -ax - \frac{m}{a} \ldots\ldots\ldots\ldots\ldots (1),$$

and to the intersecting line

$$y = -\frac{1}{a}(x - m) \ldots\ldots\ldots\ldots (2).$$

Suppose, now, that $a = -(-)^{\frac{1}{2}}\mu$, and these two equations will evidently become respectively

$$y = (-)^{\frac{1}{2}}\mu x + \frac{m}{(-)^{\frac{1}{2}}\mu} \ldots\ldots\ldots\ldots (3),$$

$$\text{and } y = \frac{1}{(-)^{\frac{1}{2}}\mu}(x - m),$$

$$\text{or } y = (-)^{\frac{1}{2}}\frac{-1}{\mu}(x - m) \ldots\ldots\ldots\ldots (4);$$

but the equations (3) and (4) are respectively the equations to a tangent to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, and to a perpendicular upon it from the focus of the branch in the plane $(+, +)$.

Hence, the equation which comprehends the representation of the former problem will likewise comprehend that of the following:

"To find the equation to the locus of the perpetual intersections of tangents to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, with perpendiculars upon them from the focus of the branch in the plane $(+, +)$."

Eliminating between (1) and (2), we get

$$y = \frac{x(x - m)}{y} + \frac{my}{x - m},$$

$$\text{or } y^2(x - m) = x(x - m)^2 + my^2,$$

and therefore $y^2 = \dfrac{x(x-m)^2}{x-2m}$,

which will comprehend both loci.

7. Various additional reflections of a like nature might be made respecting elimination between the general equations of Art. 2. for different affections of a; but what has been said above is sufficient to furnish the correct method of prosecuting such speculations.

W. W.

To the Editor of the Mathematical Journal.

SIR,—Your paper on the Extraneous Solutions of Geometrical Problems, calls my attention to a mistake which I made in an article on the Existence of Branches of Curves in various Planes. I there state (Vol. I., p. 261.) that the extraneous factor of the problem of finding the locus of the intersections of perpendiculars from the focus on the tangents to a parabola, corresponds to the intersection of perpendiculars on the tangents to the branch in the plane perpendicular to xy; whereas it really corresponds to the intersection of those tangents with lines passing through the focus, and making with the axis of x an angle complementary to that of the tangent.

The mistake arose from supposing, that if

$$y = -^{\frac{1}{2}} ax + \beta,$$

in the equation to a tangent in the plane $(+, +^{\frac{1}{4}})$; that to a line perpendicular to it is

$$y = -\frac{1}{-^{\frac{1}{2}} a}(x - m);$$

whereas it ought to be

$$y = - -^{\frac{1}{2}} \frac{1}{a}(x - m);$$

inasmuch as the $-^{\frac{1}{2}}$ has no reference to the mutual inclination of the lines, which is determined only by a. It is true that the locus of the intersections of the tangent with the perpendiculars on it from the focus does pass through the focus, as the geometrical reasoning I have employed shows; but the extraneous factor is not the equation to the locus, which your paper clearly proves. As the nature of the error is, perhaps, not very apparent, I shall feel obliged by your insertion of this correction of it.

I am your obedient servant,

Trin. Coll. D. F. GREGORY.

IX.—ON THE PERSPECTIVE OF THE CO-ORDINATE PLANES.*

I BELIEVE it will be admitted, that very few mathematical writers take any great care to make their diagrams look like what they are intended to represent. This paper will describe a simple and practicable mode of enabling the writer or the student to help himself, his readers, or both, to that clearness of conception which a well-drawn figure never fails materially to aid. The apparatus required will be a card or lamina of wood or ivory, to be presently described, and a small parallel ruler; *no compasses are required*, for ordinary purposes at least.

Let the projection be the orthographic, and let it be given that the projections of the axes make angles as follows:

$$y^{\wedge}z = \alpha, \quad z^{\wedge}x = \beta, \quad x^{\wedge}y = \gamma.$$

The most useful case being that in which the eye is situated (at an infinite distance) in that quarter of space in which all the co-ordinates are positive, I shall assume it as in the figure (see fig. 1, in No. 9, Vol. 2); α, β, and γ must then be severally greater than a right angle. Through OY and the line which (passing through O) is foreshortened into a point, draw a plane: this plane must be perpendicular to ZOX, and must cut it in a line which is projected into the continuation of OY: for as this plane passes through the eye, every angle in it is projected either into nothing or two right angles; the latter in this case, because the eye is, by hypothesis, *in* the angle. Again, any plane which contains the eye, cuts a plane parallel to that of projection in such a manner, that every line which is at right angles to their common intersection is projected into a line making a right angle with the projection of that intersection: if then we draw RP perpendicular to YO produced, we see in RP the projection of a line which is in a parallel plane to that of projection, so that RP is of the same length as its original. Repeat the process with ZO and XO, and we have the following theorem:—Every triangle PQR, which has its vertices on the projected axes, and the origin for the intersection of its *altitudes*, is either a triangle in the plane of projection, or the projection of a triangle parallel to that plane.

From this we shall readily find the projecting cosines of the actual co-ordinates: for the originals of QO, OB, are sides of a right-angled triangle, of which QB is the base, while the invisible perpendicular O′O (O′ is the origin in space) is drawn to its base. Consequently, the ratio of QO′ to its projection QO is the subduplicate ratio of QB to QO; or the co-ordinates x, y, z are foreshortened in the subduplicate ratios of PA to PO, QB to QO, and RC to RO.

* From a Correspondent.

To give an algebraical result, remember that QRP is the supplement of QOP or γ;

$$\text{whence } QB = QZ \sin \gamma.$$

$$\text{Also, } QO = QZ \sin QRO : \sin QOR,$$

$$\text{and } \sin QRO = \cos RQC = -\cos \beta;$$

$$\text{while } \sin QOR = \sin \alpha.$$

Hence the subduplicate ratio of QO : QB is

$$\sqrt{\{-\cos \beta : \sin \alpha \sin \gamma\}};$$

whence the proportions of the projections of a given line on the axes of x, y, and z, are

$$\sqrt{\left(-\frac{\cos \alpha}{\sin \beta \cdot \sin \gamma}\right)}, \quad \sqrt{\left(-\frac{\cos \beta}{\sin \gamma \cdot \sin \alpha}\right)},$$

$$\sqrt{\left(-\frac{\cos \gamma}{\sin \alpha \cdot \sin \beta}\right)} \text{ ...... (Pr. Cos.),}$$

$$\text{or } \sqrt{(-\sin 2\alpha)}, \quad \sqrt{(-\sin 2\beta)}, \quad \sqrt{(-\sin 2\gamma)},$$

which are rational, since 2α, &c., severally lie between two and four right angles.

An algebraical solution might easily be given, but it is hardly worth while. I subjoin, from one before me, the equations necessary for the solution of the inverse problem, namely, Given the projections of equal lines on the three axes, required the angles α, β, γ.

Let p, q, r be the lengths into which l is projected, according as it is parallel to x, y, or z. Then

$$l = \sqrt{\left(\frac{p^2 + q^2 + r^2}{2}\right)},$$

$$\sin \alpha = \frac{l}{bc} \sqrt{(l^2 - a^2)}, \quad \sin \beta = \frac{l}{ca} \sqrt{(l^2 - b^2)}, \quad \sin \gamma = \frac{l}{ab} \sqrt{(l^2 - c^2)}.$$

To apply this to any particular kind of projection, take a card of sufficient size, and having cut out ZOX so that $\angle ZOX = \beta$, draw OY so that $\angle XOY = \gamma$, and having slit OY, work it with a hard wood or metal point until a pencil can be made to move up and down, guided by OY, upon the paper on which the card is placed. Then calculate $\sqrt{-\sin 2\alpha}$, &c., make scales of equal parts on OX, OY, and OZ, of which the units shall be proportional to these calculated results. These scales may be repeated upon the edges, for convenience of measurement off the axes. The use of these lines is now too evident to need description, and I apprehend, that if the student were in possession of, say, three of them, laid down with different values of α, β, and γ, he would soon find he had better draw his figures for himself, than look at those which are drawn for him in the books.

A. D. M.

X.—THEOREMS IN THE CALCULUS OF GENERATING FUNCTIONS.

To the Editor of the Cambridge Journal.

SIR,—Perhaps some of your readers may take an interest in the following attempts at generalizing some of the theorems of that very graceful weapon of analysis, the Calculus of Generating Functions.

(1) Let $\phi(t)$ represent the generating function of u_x or $\phi(t) = \mathrm{G}(u_x)$.

Then, if $\psi(t)$ be any function of t capable of being expanded in integral powers of t, positive or negative, $\phi(t).\psi(t)$ will be the generating function of $\psi\left(\frac{1}{1+\Delta}\right) u_x$, or of $\psi\left(\frac{1}{\mathrm{D}}\right) u_x$, if $1 + \Delta = \mathrm{D}$.

$$\text{For let } \psi(t) = \ldots + \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2 + \ldots$$

then since $\phi(t) = u_0 + u_1 t + u_2 t^2 + \ldots\ldots$
we have for the coefficient of t^x in the product of these two series,

$$\ldots\ldots a_{-1} u_{x+1} + a_0 u_x + a_1 u_{x-1} + \ldots\ldots$$

$$= (\ldots a_{-1}\mathrm{D} + a_0 + \frac{a_1}{\mathrm{D}} + \ldots)\, u_x = \psi\left(\frac{1}{\mathrm{D}}\right) u_x,$$

$$\text{i. e. } \phi(t)\,\psi(t) = \mathrm{G}\,.\,\psi\left(\frac{1}{\mathrm{D}}\right) u_x.$$

Similarly, we may shew that

$$\phi(t)\,\psi\left(\frac{1}{t}\right) = \mathrm{G}\psi(\mathrm{D})\, u_x.$$

From these theorems we may demonstrate two problems in the Senate-House Papers for January 1840.

For, from what has been said, it follows immediately that

$$(a - \log t)^{-1}\phi(t) = \left\{a + \log\left(\frac{1}{t}\right)\right\}^{-1}\phi(t) = \mathrm{G}\,.\,(a + \log \mathrm{D})^{-1} u_x$$

$$= \mathrm{G}\,.\left(\frac{d}{dx} + a\right)^{-1} u_x, \text{ (since } \mathrm{D} = \varepsilon^{\frac{d}{dx}}),$$

$$= \mathrm{G}\,\varepsilon^{-ax}\textstyle\int \varepsilon^{ax} u_x\, dx. \text{ (See Vol. I. No. I.)}$$

This example, combined with the preceding theorem, renders it unnecessary to state that

$$\phi(t)\,\psi(\log t) = \mathrm{G}\psi\left\{\log\left(\frac{1}{\mathrm{D}}\right)\right\}.\, u_x,$$

$$\text{and } \phi(t)\,\psi\left(\log\frac{1}{t}\right) = \mathrm{G}\psi\left(\frac{d}{dx}\right).\, u_x,$$

$$\varphi(t)\,.\,\log^n\frac{1}{t} = \mathrm{G}\,\frac{d^n u_x}{dx^n}.$$

Again, for the second problem we have

$$(1 - at)^{-1}\, t\phi(t) = \mathrm{G}\,.\,(\mathrm{D} - a)^{-1}\, u_x = \mathrm{G}\,.\,a^{x-1}\,\varepsilon\,\frac{u_x}{a_x}.$$

(See Vol. I. No. II.)

Thus, to expand $(a + \log t)^{-2}\,\frac{t}{(1-t)^2}$, we may write it under the form

$$\left(a - \log\frac{1}{t}\right)^{-2}\frac{t}{(1-t)^2} = f(t), \text{ suppose.}$$

$$\text{Now, } \frac{t}{(1-t)^2} = \mathrm{G}\,.\,x;$$

$$\therefore f(t) = \mathrm{G}\left(a - \frac{d}{dx}\right)^{-2}.\,x$$

$$= \mathrm{G}\left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + c_2 x)\right\}\varepsilon^{ax}$$

$$= \frac{1}{a^2} + c_1 + \left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + c_2)\,\varepsilon^a\right\}t$$

$$+ \left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + 2c_2)\,\varepsilon^{2a}\right\}t^2 + \ldots$$

To determine the constants, put $t = 0$;

$$\therefore c_1 = -\frac{1}{a^2}.$$

Divide both sides by t, and make t again $= 0$, the value of the first side will still be 0;

$$\therefore \frac{1}{a^2}\left(1 + \frac{2}{a}\right) + \left(c_2 - \frac{1}{a^2}\right)\varepsilon^a = 0;$$

$$\therefore c_2 = \frac{1}{a^2}\left\{1 - \left(1 + \frac{2}{a}\right)\varepsilon^{-a}\right\}.$$

From these theorems we may derive the following methods of solving linear equations, either differential or of differences.

Thus, let $\psi(\mathrm{D})\,u_x = \mathrm{A}_x$ be the equation of differences. Then, if we can find $\mathrm{F}(t)$, the generating function of A_x, (which may frequently be done without difficulty,) we have, equating the generating functions on both sides,

$$\psi\left(\frac{1}{t}\right)\phi(t) = \mathrm{F}(t)$$

$$= \phi(t) = \frac{\mathrm{F}(t)}{\psi\left(\frac{1}{t}\right)},$$

and v_x = coefficient of t^x in the expansion of $\dfrac{\mathrm{F}(t)}{\psi\left(\frac{1}{t}\right)}$.

The same method applies, *mutatis mutandis*, to the differential equations, but it is in both cases difficult of application, as the coefficient of t^x in $\frac{F(t)}{\psi\left(\frac{1}{t}\right)}$ can only be obtained by the enunciation of a series whose formation is generally complicated.

E. J. L.

XI.—MATHEMATICAL NOTE.

The following is an easy method of finding the number of homogeneous products of n dimensions of m letters:—

It is evident that the number of such products is the same as the number of the terms of the n^{th} degree in the product

$$(1 + x_1 + x_1^2 + \ldots + x_1^n)(1 + x_2 + x_2^2 + \ldots + x_2^n) \ldots$$
$$\ldots (1 + x_m + x_m^2 + \ldots + x_m^n);$$

and this number is clearly equal to the coefficient of the n^{th} power of x in the development of

$$(1 + x + x^2 + \ldots + x^n)^m,$$

or, which is the same thing, in the development of

$$(1 + x + x^2 + \ldots \text{ to } \infty)^m,$$
$$\text{or of } (1 - x)^{-m},$$

which is

$$\frac{m(m+1) \ldots\ldots (m+n-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots n}.$$

ϕ.

CORRIGENDA.

In Vol. I. p. 213, line 12, should be written

$$d^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^n u}{dx^{n-1} dy} dx^{n-1} dy + \frac{n(n-1)}{n} \frac{d^n u}{dx^{n-2} dy^2} dx^{n-2} dy^2$$
$$+ \text{ \&c.}$$

In Vol. II. p. 21, line 15, *read* 'a stratum' *instead of* 'attraction.'

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I.—ON CERTAIN THEOREMS IN THE CALCULUS OF VARIATIONS.

By G. Boole, Waddington, near Lincoln.

It would perhaps have been more just to entitle this communication, "Notes on Lagrange." The papers from which it is selected were written towards the close of the year 1838, during the perusal of the *Mécanique Analytique*. Every mathematician is aware of the important uses which the illustrious author has there made of the Calculus of Variations. The mode in which it is employed as an instrument of demonstration, consists in almost every instance in a comparison of the terms of developed expressions, a mode of investigation by which the abstract existence of truths is shown, much more clearly than the nature of their mutual dependence. Now as the results obtained presuppose, in a majority of instances, the fulfilment of a certain primitive condition, (the integrability of $Xdx + Ydy + Zdz$,) it appeared to the writer of this paper, that a mode of demonstration, which should establish in a more direct way the connection between the above condition, and the great secondary principles of dynamics thence deducible, would in many respects possess the advantage. From this consideration the following attempts originated.

As the basis of these investigations, I suppose the symbols d and δ to imply two independent differentiations, performed on one variable quantity, supposed to be a function of two others. This being laid down, it is easy to establish the following principles:—

1st. The symbols, $\int$, d, δ, are mutually transposable, together with the symbols $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y}$, &c., when the denominators of these latter are relatively constant.

2nd. If U and V represent functions of u, then

$$dU\,\delta V = dV\,\delta U.$$

The last-named includes implicitly the following important cases:

1st. U being a function of u, $\dfrac{d\mathrm{U}}{du} = \dfrac{\delta \mathrm{U}}{\delta u}$.

2nd. P being a function of x, y, z,

$$\delta \mathrm{P} = \frac{d\mathrm{P}}{dx}\delta x + \frac{d\mathrm{P}}{dy}\delta y + \frac{d\mathrm{P}}{dz}\delta z.$$

The application of the Calculus of Variations to the determination of the maximum or minimum of an indeterminate integral formula $\int \mathrm{U}\,dx$, in which x is the independent variable, and U a function of x, y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, &c., presents some difficulties which are not experienced, when U is a function of x, y, dx, dy, d^2x, d^2y, &c. The reason of this appears to be, that in the former case y and its differentials are susceptible of two different kinds of variation, one resulting from x, since y is a function of that variable, the other independent of x. The former of these variations I therefore suppose to arise from some quantity entering into the constitution of x, the latter from some other quantity entering with x into the constitution of y. This distinction leads to the following very simple investigation.

Let δ_1 represent all variation through the medium of x, δ' that which is independent of x. Then

$$\delta \mathrm{U} = \frac{\delta_1 \mathrm{U}}{\delta_1 x}\delta_1 x + \frac{d\mathrm{U}}{dy}\delta' y + \frac{d\mathrm{U}}{d\frac{dy}{dx}}\delta'\frac{dy}{dx} + \&\text{c.}$$

Now since x enters into every term,

$$\frac{\delta_1 \mathrm{U}}{\delta_1 x} = \frac{1}{dx}d\mathrm{U}.$$

Moreover, by the principle of transposition,

$$\delta'\frac{dy}{dx} = \frac{d}{dx}\delta' y, \quad \delta'\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\delta' y, \ \&\text{c.}$$

Hence, on substitution,

$$\delta \mathrm{U} = \frac{1}{dx}d\mathrm{U}\,\delta x + \frac{d\mathrm{U}}{dy}\delta' y + \frac{d\mathrm{U}}{d\frac{dy}{dx}}\frac{d}{dx}\delta' y + \frac{d\mathrm{U}}{d\frac{d^2y}{dx^2}}\frac{d^2}{dx^2}\delta' y + ..$$

If by pdx we represent that portion of δy which is derived from x, then $\delta' y = \delta y - p\delta x$, which substituted in the above will give Lagrange's expression.

Every circumstance of motion of a system of free bodies in space is included in the following equation, (*Mec. Anal.*)

$$\Sigma\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z + \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z\right) = 0 \ldots (\mathrm{A}),$$

which may be resolved, in consequence of the independence of δx, δy, δz, into the triple system,

$$\frac{d^2x}{dt^2} = -X, \quad \frac{d^2y}{dt^2} = -Y, \quad \frac{d^2z}{dt^2} = -Z.$$

On changing in (A) δ into d, we have

$$\Sigma\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz + Xdx + Ydy + Zdx\right) = 0 \ldots (B);$$

and it is supposed that $Xdx + Ydy + Zdz$ is an exact differential whose integral may be represented by V.

The integration of (B) gives

$$\Sigma\left(\frac{dx^2 + dy^2 + dz^2}{2dt^2} + V\right) = H \text{ (a constant)} \ldots\ldots.. (C).$$

This equation, usually presented under the form $T + V = H$, represents the conservation of the living forces of the system.

Taking the variation of the last expression, we have

$$\Sigma\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{dt^2} + X\delta x + Y\delta y + Z\delta z\right) = 0.$$

whence, by comparison with (A),

$$\Sigma\,(dx\,d\delta x + dy\,d\delta y + dz\,d\delta z) = \Sigma\,(d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z).$$

To each side of this equation add the first side, and observing that each side becomes integrable, we shall have

$$\Sigma\delta\,(dx^2 + dy^2 + dz^2) = \Sigma d\,(dx\,\delta x + dy\,\delta y + dz\,\delta z);$$

whence

$$\delta\Sigma\int\left(\frac{dx^2 + dy^2 + dz^2}{dt}\right) = \Sigma\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right);$$

and if δx, δy, δz, be supposed to vanish for the extreme points, we have

$$\delta\int 2T\,dt = 0,$$

which expresses the principle of least action; an appellation, the propriety of which it does not fall within the scope of this paper to inquire into.

The function $\dfrac{dx^2 + dy^2 + dz^2}{2dt^2}$ being still represented by T, if for $T - V$ we substitute Z, and suppose V converted into a function of ξ, ψ, ϕ, so that T and Z become functions of ξ, ψ, ϕ, ξ', ψ', ϕ', homogeneous with respect to the three last, (since the relation between the differential coefficients of the new and old variables is linear); then will the differential equations of motion become (*Mec. Analytique*, Sec. Partie v. §. 1.)

$$\left.\begin{aligned} d\frac{dZ}{d\xi'} - \frac{dZ}{d\xi}dt &= 0, \\ d\frac{dZ}{d\psi'} - \frac{dZ}{d\psi}dt &= 0, \\ d\frac{dZ}{d\phi'} - \frac{dZ}{d\phi}dt &= 0, \end{aligned}\right\} \ldots.. (D).$$

I shall now proceed to demonstrate a theorem, forming the basis of Lagrange's investigations on the great problem of the variation of the arbitrary constants, in questions of dynamics, using, as he has done, the symbol d to denote differentiation with respect to the time, and Δ and δ to imply variations relative to the arbitrary constants which may be supposed to enter into the constitution of ξ, ψ, ϕ.

Since Z is a function of ξ, ψ, ϕ, ξ', ψ', φ',

$$\delta Z = \frac{dZ}{d\xi}\,\delta\xi + \frac{dZ}{d\xi'}\,\delta\xi' + (\quad),$$

$$\Delta Z = \frac{dZ}{d\xi}\,\Delta\xi + \frac{dZ}{d\xi'}\,\Delta\xi' + (\quad),$$

( ) indicating the corresponding terms with respect to ψ, ψ', φ, φ'. Now $\Delta\delta Z = \delta\Delta Z$; whence, performing the requisite operations, and transposing to the first side,

$$\left.\begin{array}{l} \Delta\,\frac{dZ}{d\xi}\,\delta\xi + \frac{dZ}{d\xi}\,\Delta\delta\xi + \Delta\,\frac{dZ}{d\xi'}\,\delta\xi' + \frac{dZ}{d\xi'}\,\Delta\delta\xi' + (\;) \\ -\,\delta\,\frac{dZ}{d\xi}\,\Delta\xi - \frac{dZ}{d\xi}\,\delta\Delta\xi - \delta\,\frac{dZ}{d\xi'}\,\Delta\xi' - \frac{dZ}{d\xi'}\,\delta\Delta\xi' - (\;) \end{array}\right\} = 0.$$

The even terms of this expression annul each other.

Moreover by (D), $\frac{dZ}{d\xi} = \frac{d}{dt}\,\frac{dZ}{d\xi'}$; and by the principle of transposition laid down at the commencement of this paper,

$$\delta\xi' = \delta\,\frac{d\xi}{dt} = \frac{d}{dt}\,\delta\xi, \quad \text{and} \quad \Delta\xi' = \frac{d}{dt}\,\Delta\xi.$$

Making these substitutions, and rejecting the common denominator dt, we have

$$\left.\begin{array}{l} d\Delta\,\frac{dZ}{d\xi'}\,\delta\xi + \Delta\,\frac{dZ}{d\xi'}\,d\delta\xi + (\;) \\ -\,d\delta\,\frac{dZ}{d\xi'}\,\Delta\xi - \delta\,\frac{dZ}{d\xi'}\,d\Delta\xi - (\;) \end{array}\right\} = 0.$$

This equation is integrable, and gives

$$\Delta\,\frac{dZ}{d\xi'}\,-\delta\xi\;\delta\,\frac{dZ}{d\xi'}\,\Delta\xi + (\;) = C.$$

Now since ξ', ψ', ϕ', do not enter into V, $\frac{dZ}{d\xi'} = \frac{dT}{d\xi'}$. Making this substitution, and supplying the deficient terms, we arrive at the theorem sought,

$$\left.\begin{array}{l} \Delta\,\frac{dT}{d\xi'}\,\delta\xi + \Delta\,\frac{dT}{d\psi'}\,\delta\psi + \Delta\,\frac{dT}{d\phi'}\,\delta\phi \\ -\,\delta\,\frac{dT}{d\xi'}\,\Delta\xi - \delta\,\frac{dT}{d\psi'}\,\Delta\psi - \delta\,\frac{dT}{d\varphi'}\,\Delta\varphi \end{array}\right\} = C.$$

The investigation of this theorem in the *Mec. Analytique* occupies nearly four pages, owing to the lengthened developments which are there introduced.

I shall now proceed to demonstrate from the general transformed equation of motion, the principles of the conservation of living forces, and of least action. The former of these has been thence deduced by Lagrange. I am not however aware that the latter has been obtained from the same equation, either by the discoverer of the Calculus of Variations, or by any subsequent author.

The general differential equation being written under the form

$$\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}\right)\delta\xi+\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\psi'}-\frac{d\mathrm{T}}{d\psi}\right)\delta\psi+\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\phi'}-\frac{d\mathrm{T}}{d\phi}\right)\delta\phi+\delta\mathrm{V}=0 \;(\mathrm{A}),$$

we have, on changing δ into d,

$$\left(\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}\right)d\xi+(\quad)+d\mathrm{V}=0 \;\ldots\ldots\ldots (\mathrm{B}).$$

Integrating, we obtain

$$\frac{d\mathrm{T}}{d\xi'}\xi'+\frac{d\mathrm{T}}{d\psi'}\psi'+\frac{d\mathrm{T}}{d\phi'}\phi'-\mathrm{T}+\mathrm{V}=\mathrm{H} \text{ (a constant)}\ldots (\mathrm{C});$$

which, by the theorem for homogeneous functions, becomes

$$\mathrm{T}+\mathrm{V}=\mathrm{H},$$

and expounds the principle of the conservation of living forces.

Take now the variation of (C),

$$\left.\begin{array}{l}\delta\dfrac{d\mathrm{T}}{d\xi'}\xi'+\dfrac{d\mathrm{T}}{d\xi'}\delta\xi'+(\quad)\\ -\dfrac{d\mathrm{T}}{d\xi}\delta\xi-\dfrac{d\mathrm{T}}{d\xi'}\delta\xi'-(\quad)\end{array}\right\}+\delta\mathrm{V}=0.$$

The even terms mutually destroying each other, we have, on equating the remaining ones with the first side of (A),

$$\delta\frac{d\mathrm{T}}{d\xi'}\xi'-\frac{d\mathrm{T}}{d\xi}\delta\xi+(\quad)+\delta\mathrm{V}=\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}\delta\xi-\frac{d\mathrm{T}}{d\xi}\delta\xi+(\quad)+\delta\mathrm{V};$$

$$\text{therefore}\quad \delta\frac{d\mathrm{T}}{d\xi'}\xi'+(\quad)=\frac{d}{dt}\frac{d\mathrm{T}}{d\xi'}\delta\xi+(\quad).$$

To this add the identical equation,

$$\frac{d\mathrm{T}}{d\xi'}\delta\xi'=\frac{d\mathrm{T}}{d\xi'}\frac{d}{dt}\delta\xi;$$

and observing that each side becomes integrable, we have, on performing the integrations and supplying the deficient terms,

$$\delta\left(\frac{d\mathrm{T}}{d\xi'}\xi'+\frac{d\mathrm{T}}{d\psi'}\psi'+\frac{d\mathrm{T}}{d\phi'}\phi'\right)=\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\xi'}\delta\xi+\frac{d\mathrm{T}}{d\psi'}\delta\psi+\frac{d\mathrm{T}}{d\phi'}\delta\phi\right).$$

The first side of the above equation is, by the theorem for homogeneous functions of the second degree, equal to $\delta(2\mathrm{T})$.

Hence, if $\delta\xi$, $\delta\psi$, $\delta\phi$, are supposed to vanish for the extreme points, we have

$$\delta\int 2\text{T}\,dt = 0,$$

the same result as was obtained in the former investigation, from the equation in rectangular co-ordinates.

II.—ON AN EXPRESSION FOR ANY POSITIVE INTEGRAL POWER OF A LOGARITHM.

The peculiar method by which Lagrange, in the *Théorie des Fonctions*, has arrived at the expansion of $\log_a x$, we propose to apply to the determination of an expression for $(\log_a x)^{m+1}$, where m is any positive integer.

Assume $a^y = x$:

then clearly

$$\{1 + (a-1)\}^y = 1 + (x-1),$$

and therefore

$$\{1 + (a-1)\}^{ny} = \{1 + (x-1)\}^n,$$

where n is any quantity whatever.

Hence, by the binomial theorem,

$$1 + ny(a-1) + \frac{ny(ny-1)}{1.2}(a-1)^2 + \frac{ny(ny-1)(ny-2)}{1.2.3}(a-1)^3 + \ldots$$

$$= 1 + n(x-1) + \frac{n(n-1)}{1.2}(x-1)^2 + \frac{n(n-1)(n-2)}{1.2.3}(x-1)^3 + \ldots$$

Suppose now that

${}^{m+1}\text{S}_1$ = the sum of the numbers 1, 2, 3, ... $m+1$,

${}^{m+2}\text{S}_2$ = the sum of the products of every two of 1, 2, 3,...$m+2$,

${}^{m+3}\text{S}_3$ = ... three of 1, 2, 3,...$m+3$,

&c. = &c.

Then clearly the coefficient of n^{m+1}, if m be a positive integer, will, in the former member of the equation, be

$$y^{m+1}.\left\{\frac{(a-1)^{m+1}}{1.2.3..(m+1)} - {}^{m+1}\text{S}_1.\frac{(a-1)^{m+2}}{1.2.3..(m+2)} + {}^{m+2}\text{S}_2.\frac{(a-1)^{m+3}}{1.2.3..(m+3)} - ..\right\}$$

and in the latter

$$\frac{(x-1)^{m+1}}{1.2.3...(m+1)} - {}^{m+1}\text{S}_1.\frac{(x-1)^{m+2}}{1.2.3...(m+2)} + {}^{m+2}\text{S}_2.\frac{(x-1)^{m+3}}{1.2.3...(m+3)} - ..$$

and therefore, equating these two expressions, which, n being a

perfectly arbitrary quantity, we are at liberty to do by the theory of indeterminate coefficients, we have

$$y^{m+1} \text{ or } (\log_a x)^{m+1} = \frac{\dfrac{(x-1)^{m+1}}{1.2.3...(m+1)} - {}^{m+1}S_1 . \dfrac{(x-1)^{m+2}}{1.2.3...(m+2)} + {}^{m+2}S_2 . \dfrac{(x-1)^{m+3}}{1.2.3...(m+3)} - ..}{\dfrac{(a-1)^{m+1}}{1.2.3..(m+1)} - {}^{m+1}S_1 . \dfrac{(a-1)^{m+2}}{1.2.3...(m+2)} + {}^{m+2}S_2 . \dfrac{(a-1)^{m+3}}{1.2.3...(m+3)} - ..}$$

If we put $m = 0$, this formula is evidently reduced to

$$\log_a x = \frac{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - ...}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - ...}.$$

which is the ordinary expression for $\log_a x$.

W. W.

III.—ON THE GENERAL INTERPRETATION OF EQUATIONS BETWEEN TWO VARIABLES IN ALGEBRAIC GEOMETRY.

By W. WALTON, B.A., Trinity College.

1. THE object of the present paper is to discuss the general geometrical signification of an equation involving two variables, x and y. The propositions which we propose to establish are the two following:—first, that when both the magnitudes and the affections of x and y experience simultaneously every conceivable variation consistently with their mutual relation, the equation will represent a curve surface; and, secondly, that when unlimited variations of magnitude are assigned to x and y, mutually consistent, while the affection of either of them is restricted to a constant state, the equation will represent a curve line, there being a different curve line for each different constant state of the affection of the restricted variable, and that every one of this series of curve lines lies in the surface of our first proposition.

In the demonstration of these two propositions, we have in the first instance employed the symbol of revolution $+^p$ for the construction of the indefinite number of pairs of axes of x and y, in a manner different from that which has been adopted by Mr. Gregory in his article on the Existence of Branches of Curves in several Planes, (see Vol. I. No. VI. of this Journal), the only principle by which we consider that we are necessarily to be guided in this matter being, that supposing $+^r a$ and $+^s\beta$ to be corresponding values of x and y, the corresponding conjugate axes of x and y shall make with the original axes the angles $2r\pi$ and $2s\pi$ respectively, their position being additionally and finally defined by any

fixed law to which we may choose to subject them. We have also, in the second instance, shewn that, according either to the method developed by Mr. Gregory, or to any other admissible method of constructing the pairs of conjugate axes, the same two propositions are equally true.

In the investigations upon which we are about to enter, we shall explain the method of translating the expressions both for the curves and for the surface, from the affectional equation between the two variables x and y, to equivalent quantitive equations between three variables, x, y, and z.

2. Let the general equation between two variables, x and y, be represented by

$$f(x, y, A_1, A_2, A_3, \dots A_\tau) = 0 \dots\dots\dots (1),$$

where f is a general symbol of functionality, and A_1, A_2, A_3, ... A_τ are the arbitrary constants of the equation.

$$\begin{aligned} \text{Let }\; A_1 &= \phi_1(+, -) \,.\, k_1, \\ A_2 &= \phi_2(+, -) \,.\, k_2, \\ \&c. &= \&c. \\ A_\tau &= \phi_\tau(+, -) \,.\, k_\tau, \end{aligned}$$

where k_1, k_2, k_3, ... k_τ are merely symbols of quantity, and $\phi_1(+, -)$, $\phi_2(+, -)$, ... $\phi_\tau(+, -)$ any functions whatever of $+$ and $-$. These it is evident are the most general forms for the arbitrary constants of which we can form any conception. Now we know that

$$+ = \cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi,$$

$$\text{and } - = \cos \pi + (-)^{\frac{1}{2}} \sin \pi,$$

and therefore

$$\phi_1(+, -) = \phi_1 \{\cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi,\ \cos \pi + (-)^{\frac{1}{2}} \sin \pi\},$$

and similarly for the expressions for A_2, A_3, .. A_τ.

But it is clear that in all cases the expression for $\phi_1(+, -)$ can be expanded by some algebraical process, so as to give a result of the form

$$M + (-)^{\frac{1}{2}} N,$$

where M and N are symbols of quantity.

$$\begin{aligned} \text{Hence we have }\quad A_1 &= \phi_1(+, -) \,.\, k_1, \\ &= Mk_1 + (-)^{\frac{1}{2}} Nk_1. \end{aligned}$$

Put $Mk_1 = a_1 \cos 2m_1\pi$ and $Nk_1 = a_1 \sin 2m_1\pi$, and we have

$$\begin{aligned} A_1 &= a_1 \{\cos 2m_1\pi + (-)^{\frac{1}{2}} \sin 2m_1\pi\}, \\ &= \{\cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi\}^{m_1} \,.\, a_1, \\ &= +^{m_1} a_1. \end{aligned}$$

In the same way we may shew, that

$$\begin{aligned} A_2 &= +^{m_2} . a_2, \\ A_3 &= +^{m_3} . a_3, \\ \&c. &= \&c. \\ A_T &= +^{m_T} . a_T. \end{aligned}$$

The equation (1) then, when the affections of its parameters are made explicit, becomes

$$f\{x, y, \ +^{m_1}a_1, \ +^{m_2}a_2, \ +^{m_3}a_3, \ldots +^{m_T}a_T\} = 0.$$

Again, the general expression for the variable value of x is, as we may shew by analogous reasoning, $+^r a$, where r and a are variable quantities, whose variations are mutually independent, and where we suppose a to be a symbol of magnitude alone, $+^r$ being the representative of the affection of position. Let $+^s\beta$ be the corresponding value for y.

Hence, the equation (1) is equivalent to

$$f\{+^r a, \ +^s\beta, \ +^{m_1}a_1, \ +^{m_2}a_2, \ +^{m_3}a_3, \ldots\ldots +^{m_T}a_T\} = 0;$$

and if we put

$$\begin{aligned} +^r &= \cos 2r\pi + (-)^{\frac{1}{2}} \sin 2r\pi, \\ +^s &= \cos 2s\pi + (-)^{\frac{1}{2}} \sin 2s\pi, \\ +^{m_1} &= \cos 2m_1\pi + (-)^{\frac{1}{2}} \sin 2m_1\pi, \\ \&c. &= \&c \end{aligned}$$

and then expand by some algebraical process the former member of this equation, we shall obviously get an equation of the form

$$\phi(a, \beta, r, s) + (-)^{\frac{1}{2}} \psi(a, \beta, r, s) = 0,$$

where ϕ and ψ are symbols of functionality; and this equation clearly resolves itself into the two following:

$$\phi(a, \beta, r, s) = 0 \ \ldots\ldots\ldots\ldots\ldots \ (2),$$

$$\text{and } \psi(a, \beta, r, s) = 0 \ \ldots\ldots\ldots\ldots\ldots \ (3).$$

3. Let Ox, Oy, be the axes of x and y, and from O draw Oz perpendicular to the plane of x, y. In the plane xOz draw OE equal to a, and making an angle $2r\pi$ with the axis of x. And in the plane yOz draw OF equal to β, and making an angle $2s\pi$ with the axis of y. From E draw EP parallel to OF, and meeting FP drawn from F parallel to OE in the point P. Then is the point P a point in the locus of the equation (1).

From E and F draw EM and FN, meeting respectively in M and N, the axes of x and y at right angles, and from P let fall PQ meeting at right angles in the point Q the plane of x, y. Also from E draw ER parallel to MQ, and meeting PQ in R. Then clearly RQ is equal to EM; and since EP, ER, are evidently equal

P

respectively to OF and ON and the angle PER to the angle FON, it is clear that PR is equal to FN.

Let OM, ON, PQ be called respectively x', y', z'. Then clearly, from what has been said, we have

$$x' = \alpha \cos 2r\pi \ldots\ldots\ldots\ldots(4)$$
$$y' = \beta \cos 2s\pi \ldots\ldots\ldots\ldots(5)$$
$$\text{and } z' = \alpha \sin 2r\pi + \beta \sin 2s\pi \ldots\ldots\ldots(6).$$

4. If between the five equations (2), (3), (4), (5), (6), we eliminate the four quantities α, β, r, s, we shall evidently obtain an equation

$$\theta\,(x', y', z') = 0 \ldots\ldots\ldots\ldots(7),$$

where $\theta\,(x', y', z')$ represents some function of x', y', z'.

Thus we see that the equation (1), when interpreted according to the utmost generality of the meanings of x and y, denotes a surface represented quantitively by the equation (7).

5. If instead of supposing r a variable quantity, we consider it to be constant, we shall get as the result of eliminating α, β, and s between the five equations (2), (3), (4), (5), (6), the two following equations:

$$\chi\,(x', y', z') = 0 \ldots\ldots\ldots\ldots(8)$$
$$\omega\,(x', y', z') = 0 \ldots\ldots\ldots\ldots(9),$$

where $\chi\,(x', y', z')$, $\omega\,(x', y', z')$, represent certain functions of x', y', z'. Thus we see that if the affection of x in the equation (1) remain invariable, this equation will represent a curve line whose equations in x', y', z' are (8) and (9).

6. Since from what has been said, it is evident that the equation (7) must result from the elimination of r between the equations (8) and (9), it is clear that the equation (7) must correspond to a surface formed by a series of curves represented by the equations (8) and (9) when r experiences every degree of variation. Hence, conversely, every curve of the class represented by the equations (8) and (9) must lie altogether within the surface represented by the equation (7).

7. In illustration of the principles which we have developed above, we will furnish one or two examples. Let us take as a first example the equation

$$x^2 + y^2 = A^2.$$

Assuming

$$x = +\,^{r}\alpha,\ y = +\,^{s}\beta,\ k = +\,^{m}b,$$

we have

$$+\,^{2r}\alpha^2 + +\,^{2s}\beta^2 = +\,^{2m}b^2.$$

Hence putting for $+$ its trigonometrical value, and equating the coefficient of $(-)^{\frac{1}{2}}$ and the part which is independent of this symbol separately to zero, we have

$$\alpha^2 \,.\, \cos 4r\pi + \beta^2 \,.\, \cos 4s\pi = b^2 \,.\, \cos 4m\pi,$$
$$\text{and } \alpha^2 \,.\, \sin 4r\pi + \beta^2 \,.\, \sin 4s\pi = b^2 \,.\, \sin 4m\pi.$$

From these two equations and the equations (4), (5), (6), we have, by the elimination of a and β, the following equations:

$$x'^2 \cdot \frac{\cos 4r\pi}{\cos^2 2r\pi} + y'^2 \cdot \frac{\cos 4s\pi}{\cos^2 2s\pi} = b^2 \cdot \cos 4m\pi$$

$$x'^2 \cdot \frac{\sin 4r\pi}{\cos^2 2r\pi} + y'^2 \cdot \frac{\sin 4s\pi}{\cos^2 2s\pi} = b^2 \cdot \sin 4m\pi$$

$$z' = x' \cdot \tan 2r\pi + y' \cdot \tan 2s\pi.$$

Now the two former of these equations may be written

$$x'^2 + y'^2 - x'^2 \tan^2 2r\pi - y'^2 \tan^2 2s\pi = b^2 \cdot \cos 4m\pi$$

$$2x'^2 \cdot \tan 2r\pi + 2y'^2 \cdot \tan 2s\pi = b^2 \cdot \sin 4m\pi\,;$$

and, therefore, by virtue of the third of them, we have the two following equations:

$$x'^2 + y'^2 - x'^2 \tan^2 2r\pi - (z' - x' \tan 2r\pi)^2 = b^2 \cdot \cos 4m\pi,$$

and

$$2x'(x' - y') \tan 2r\pi + 2y'z' = b^2 \cdot \sin 4m\pi.$$

If r be considered of invariable magnitude, these two equations represent a curve line. But if r be reckoned variable, we have, eliminating it between them, the following result:

$$4(x' - y')^2(x'^2 + y'^2) = 4b^2 . (x' - y')^2 . \cos 4m\pi + (2x'z' - b^2 \sin 4m\pi)^2$$
$$+ (2y'z' - b^2 \sin 4m\pi)^2,$$

which is the equation to the surface represented by the affectional equation

$$x^2 + y^2 = A^2.$$

8. As a second example we will take the equation

$$y^2 = Ax.$$

Assuming $x = +^{r} a$, $y = +^{s}\beta$, $A = +^{m} b$, we have

$$+^{2s}\beta^2 = +^{r+m} ba,$$

whence

$$\{\cos 4s\pi + (-)^{\frac{1}{2}} \sin 4s\pi\} . \beta^2$$
$$= \{\cos 2(r+m)\pi + (-)^{\frac{1}{2}} \sin 2(r+m)\pi\}\, ba\,;$$

and, therefore,

$$\beta^2 \cos 4s\pi = ba \cos 2\,(r+m)\,\pi,$$

$$\text{and } \beta^2 \sin 4s\pi = ba \sin 2\,(r+m)\,\pi.$$

Hence we have

$$\beta^2 = ba,$$

$$\text{and } 2s\pi = (r+m)\,\pi\,;$$

and therefore by the equations (4) and (5), we have

$$bx' = \beta^2 \cos 2r\pi$$

$$y'^2 = \beta^2 \cos^2 2s\pi\,;$$

and, therefore,

$$bx' \cos^2 2s\pi = y'^2 . \cos^2 2r\pi;$$

and, therefore,

$$bx' \cos^2 (r+m)\pi = y'^2 . \cos^2 2r\pi;$$

and from the equations (4), (5), (6),

$$z' = x' \tan 2r\pi + y' \tan 2s\pi,$$

$$\text{or } z' = x' \tan 2r\pi + y' \tan (r+m)\pi.$$

If r be invariable, then these two last equations represent a curve line. And supposing r to be variable, we shall obtain for the locus of the equation $y^2 = Ax$ the appropriate equation in x', y', z', by eliminating the r between the two equations to the curve line.

If we put $r=0$, and $m=0$, we get for the equations to the curve

$$bx' = y'^2$$

$$z' = 0,$$

which are the equations to the common parabola in the plane x', y'.

9. As a third example we will take the equation of the first degree in x and y.

$$Ax + By = 1 \text{..... ......(A)}.$$

Assume $A = +^{m}a$, $B = +^{n}b$, $x = +^{r}\alpha$, and $y = +^{s}\beta$, and this equation assumes the form

$$+^{m+r} a\alpha + +^{n+s} b\beta = 1.$$

Hence, putting

$$+ = \cos 2\pi + (-)^{\frac{1}{2}} \sin 2\pi,$$

we have

$$a\alpha \cos. 2(m+r)\pi + b\beta \cos. 2(n+s)\pi + (-)^{\frac{1}{2}} \{a\alpha \sin. 2(m+r)\pi + b\beta \sin. 2(n+s)\pi\} = 1,$$

and this equation resolves itself into the two following:

$$a\alpha \cos. 2(m+r)\pi + b\beta \cos. 2(n+s)\pi = 1 \text{......(B)}$$

$$\text{and } a\alpha \sin. 2(m+r)\pi + b\beta \sin. 2(n+s)\pi = 0 \text{......(C)}.$$

Multiplying the former and the latter of these equations respectively by $\cos 2n\pi$ and $\sin 2n\pi$, and adding the results, we get

$$a\alpha \cos. 2(m+r-n)\pi + b\beta \cos 2s\pi = \cos 2n\pi \text{......(D)},$$

and multiplying them by $\cos 2m\pi$ and $\sin 2m\pi$ respectively, and adding the results, we have

$$a\alpha \cos 2r\pi + b\beta \cos 2(n+s-m)\pi = \cos 2m\pi \text{..... (E)}.$$

Again, multiplying the equations (4), (5), (6), respectively by the three following expressions:

$a \{b \cos 2(n-m)\pi - a\}$, $b \{b - a \cos 2(n-m)\pi\} y'$, $ab \sin 2(n-m)\pi$,

and adding the results, we have for the left-hand member of the resulting equation

$$a\,\{b\cos 2(n-m)\pi - a\}\,x' + b\,\{b - a\cos 2(n-m)\pi\}\,y' + ab\sin 2(n-m)\pi\,.\,z',$$

and for the right-hand member

$$\{ab\cos 2(n-m)\pi - a^2\}\,\alpha\cos 2r\pi + \{b^2 - ab\cos 2(n-m)\pi\}\,\beta\cos 2s\pi + ab\alpha\sin 2(n-m)\pi\,.\,\sin 2r\pi + ab\beta\sin 2(n-m)\pi\,.\,\sin 2s\pi,$$

or

$$b\,\{a\alpha\cos 2(m+r-n)\pi + b\beta\cos 2s\pi\} - a\,\{a\alpha\cos 2r\pi + b\beta\cos 2(n+s-m)\pi\};$$

and, therefore, by virtue of the equations (D) and (E),

$$b\cos 2n\pi - a\cos 2m\pi.$$

Hence the affectional equation (A) is equivalent to the following quantitive one,

$$a\,\{b\cos 2(n-m)\pi - a\}\,x' + b\,\{b - a\cos 2(n-m)\pi\}\,y' + ab\sin 2(n-m)\pi\,.\,z' = b\cos 2n\pi - a\cos 2m\pi;$$

or, if symmetry of expression be considered necessary,

$$2a\,\{b\cos 2(m-n)\pi - a\}\,x' + ab\sin 2(n-m)\pi\,.\,z' + 2a\cos 2m\pi = 2b\,\{a\cos 2(n-m)\pi - b\}\,y' + ba\sin 2(m-n)\pi\,.\,z' + 2b\cos 2n\pi \ldots\ldots(\mathrm{F}),$$

which is an equation of the first degree in x', y', z'.

Hence we see that the equation of the first degree between two variables represents a plane surface, when the utmost generality of signification and of variation is assigned to its elemental letters.

10. Let the equation to the plane surface which is represented by the equation (F) be conceived to be expressed under the form

$$\frac{x'}{\beta_1} + \frac{y'}{\beta_2} + \frac{z'}{\beta_3} = 1 \ldots\ldots\ldots\ldots(\mathrm{G}).$$

where β_1, β_2, β_3, are the parameters of the equation.

Then comparing this equation with the equation (F), we have the three following relations:

$$a\beta_1\,\{b\cos 2(n-m)\pi - a\} = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots(\mathrm{H})$$

$$b\beta_2\,\{b - a\cos 2(n-m)\pi\} = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots(\mathrm{I})$$

$$\text{and } ab\beta_3\sin 2(n-m)\pi = b\cos 2n\pi - a\cos 2m\pi \ldots\ldots(\mathrm{K}).$$

Now it is evidently sufficient and necessary for the conservation of the identity of the plane (G), and therefore of the plane (F), with which it is coincident, that the values of the quantities β_1, β_2, β_3, experience separately no variation. Hence, clearly, so long as the three relations (H), (I), (K), are satisfied for constant values of

these quantities, the identity of the locus of the affectional equation (A) will be secured. But it is obvious that since the three equations (H), (I), (K), involve four quantities a, b, m, n, the values of these quantities must remain individually indeterminate. Hence we see that it is not necessary, as we have been supposing, that the quantities a, b, m, n, should remain constant in order to secure the identity of the locus of the equation (A), but merely that they preserve always the mutual relations which we have explained.

11. If between the five equations (D), (E), (H), (I), (K), we eliminate successively a, b, s, n, and a, b, β, n, we shall evidently get respectively

$$\beta = \phi\,(\alpha, r, m)$$
$$\text{and } s = \chi\,(\alpha, r, m),$$

where $\phi\,(\alpha, r, m)$ represents a function of α, r, m, not involving any of the quantities a, b, s, n, and $\chi\,(\alpha, r, m)$ a function of α, r, m, not involving any of the quantities a, b, β, n.

From these expressions for β and s, it appears that for any assigned values of α and r, β and s will experience an infinite variety of values with the variation of the value of m; and by what has been said in the preceding section, it is clear that, without affecting the identity of the locus of the equation (A), we may assign to m whatever values we please. This amounts to saying, that having taken a certain angle E$\mathrm{O}x$, and a certain length OE, the angle F$\mathrm{O}y$ and the length OF will not be defined in magnitude, but that without affecting the identity of the plane surface represented by the equation (A), an infinite variety of points P in it will be determinable for any assigned magnitudes of the angle E$\mathrm{O}x$, and the length OE, according to the value which we assign to m. Hence we see that the equation (A) not only represents a definite plane, but likewise furnishes us with an infinite number of ways of arriving geometrically at each of its constituent points.

12. If we wish to transform the equation to any plane from the quantitive shape (G) to the affectional shape (A), it is plain that from the three equations (H), (I), (K), we must obtain for a, b, and n, expressions in terms of β_1, β_2, β_3, and m, and we shall have for the resulting equation a form

$$+^{m}\phi(\beta_1, \beta_2, \beta_3, m)\,.\,x + +^{\psi(\beta_1, \beta_2, \beta_3, m)}.\chi(\beta_1, \beta_2, \beta_3, m)\,.\,y = 1,$$

the locus of which will be the same whatever value we may assign to m.

13. If between the three equations (H), (I), (K), we eliminate a and b, we shall ultimately arrive at the expression

$$(\beta_1 - \beta_2)\,\beta_3 = (\beta_1\beta_2 + \beta_3{}^2)\tan 2\,(n - m)\,\pi,$$

a result which shews that the value of $n - m$ must always remain constant for the conservation of the identity of the locus of the equation (A).

We may also easily arrive at the following relation from the same three equations

$$\frac{b}{a}-\frac{a}{b}=\frac{(\beta_1+\beta_2)\beta_3}{\beta_1\beta_2}\sin 2(n-m)\pi,$$

which, in conjunction with the previous result, shews that the relation between a and b must be invariable.

14. Again, suppose that r is constant, or that the affections of x in the equation (A) are restricted.

Then multiplying the equation (D) by $\cos 2r\pi$, and introducing the relations afforded by the equations (4) and (5), we have

$$ax'\cos 2(m-n+r)\pi+by'\cos 2r\pi=\cos 2n\pi\,.\cos 2r\pi\ldots(a).$$

and multiplying the equation (E) by $\cos 2s\pi$, we have in like manner

$$ax'\cos 2s\pi+by'\cos 2(n-m+s)\pi=\cos 2m\pi\,.\cos 2s\pi\ldots(b),$$

and from the three equations (4), (5), (6), we have

$$z'=x'\tan 2r\pi+y'\tan 2s\pi,$$

and between this last equation and the equation (b), we can readily arrive at the equation

$$\{a-b\tan 2r\pi\,.\sin 2(m-n)\pi\}\,.\,x'+b\cos 2(m-n)\pi\,.\,y'$$
$$+\,bz'\sin 2(m-n)\pi=\cos 2m\pi\ldots\ldots(c).$$

Hence we see that when x is restricted in point of its affections, the equation (A) represents a straight line whose equations are (a) and (c) expressed quantitively.

15. If m, n, r, be all put equal to zero, the equations (B) and (C) become

$$a\alpha+b\beta\cos 2s\pi=1$$
$$b\beta\sin 2s\pi=0,$$

whence $s=0$, and therefore,

$$a\alpha+b\beta=1.$$

Hence from (4), (5), (6), we have

$$x'=\alpha,\quad y'=\beta,\quad z'=0,$$

and, therefore, we get

$$ax'+by'=1$$
$$z'=0,$$

as the equations to the straight line represented by the equation (A).

16. We will take a single example of the case in which the values of x in the equation (A) are restricted to mere magnitude.

Let the equation be

$$y=+^{\frac{1}{2}}k(x-l).$$

Put the equation

$$+^{m} ax + +^{n} by = 1,$$

under the form

$$y = - +^{m-n} \frac{a}{b}\left(x - +^{-m} \frac{1}{a}\right),$$

and comparing the coefficients of like terms in the two, we have

$$l = +^{-m} \frac{1}{a}, \text{ whence } m = 0 \text{ and } a = \frac{1}{l},$$

$$\text{and } +^{\frac{1}{4}} k = - +^{m-n} \frac{a}{b},$$

$$\text{whence } n = - \frac{1}{4} \text{ and } b = - \frac{1}{lk}.$$

Hence, by substituting these values in the equations (a) and (c) and putting $r = 0$, we have

$$y' = 0,$$

$$\text{and } ax' + bz' = 1,$$

$$\text{or } \frac{1}{l} x' - \frac{1}{kl} z' = 1,$$

$$\text{or } z' = k(x' - l),$$

which are the two equations to the line.

Thus we see that prefixing the symbol $+^{\frac{1}{4}}$ to k in the equation

$$y = k(x - l),$$

is equivalent to turning the corresponding line into a plane at right angles to that of x, y, without altering its inclination to the axis of x, or the point in which it cuts it.

17. We will now proceed to shew that the two propositions, which it has been the object of the present paper to establish, hold true likewise, when, instead of the application which we have chosen of the symbol of revolution $+^{p}$, we adopt the application developed in Mr. Gregory's article.

In fact, suppose as before that

$$x = +^{r} \alpha \text{ and } y = +^{s} \beta.$$

Transfer the axis Ox through an angle $2r\pi$ in a plane perpendicular to Oy into a new axis Ox_1, and then turn Oy through an angle $2s\pi$ in a plane perpendicular to the axis Ox_1 into an axis Oy_1. Measure a length α along Ox_1, and a length β along Oy_1; and then in the new system of axes Ox_1 and Oy_1, consider α and β as the co-ordinates of the point defined by the equations $x = +^{r} \alpha$ and $y = +^{s} \beta$.

Let x', y', z', represent the quantitive co-ordinates of this same point. Then, by Euler's formulæ for transformation from one system of rectangular co-ordinates to another likewise rectangular, we

shall have, recollecting that in the new system the co-ordinate perpendicular to the plane of x_1Oy_1 is put equal to zero,

$$x' = a\cos 2r\pi - \beta\sin 2r\pi\,.\sin 2s\pi,$$
$$y' = \beta\cos 2s\pi,$$
$$z' = a\sin 2r\pi + \beta\cos 2r\pi\,.\sin 2s\pi;$$

and dealing with these three equations in the place of the equations (4), (5), (6), we shall, by exactly the same kind of reasoning as we have employed in the investigations of the preceding sections, be enabled to determine the curve lines and the curve surface belonging to the equation $f(x, y) = 0$, and to shew that the curve lines all lie within the curve surface.

And, generally, if we were to select any other admissible law for the determination of the indefinite number of pairs of conjugate axes, and that too whether the prime axes be rectangular or oblique, we should evidently obtain for x', y', z', expressions functional of a, β, r, s, and analogous conclusions in respect to the curve lines and the curve surface would obviously be established.

IV.—VARIATION OF NODE AND INCLINATION.

The following method of finding the variations of the inclination and longitude of the node, is more convenient than that given in Pratt's *Mechanical Philosophy*, p. 336.

Adopting the notation usual in the lunar theory, we have

$$s = k\sin(\theta - \gamma)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{also}\quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0\ldots\ldots\ldots\ldots(2).$$

In the disturbed orbit, (1) and its first derived equation will be true, as if the elements were invariable; which gives the equation

$$\sin(\theta - \gamma)\frac{dk}{dt} - k\cos(\theta - \gamma)\frac{d\gamma}{dt} = 0\ldots\ldots(3),$$

and differentiating (1) a second time, there is

$$\frac{d^2s}{dt^2} = k\frac{d\theta}{dt}\cdot\frac{d}{d\theta}\left(\cos(\theta - \gamma)\frac{d\theta}{dt}\right)$$
$$+\left(\cos(\theta - \gamma)\frac{dk}{dt} + k\sin(\theta - \gamma)\frac{d\gamma}{dt}\right)\frac{d\theta}{dt}.$$

The second and third terms are those due to perturbation. Also, the inclination being very small, the effect of perturbation on $\frac{d^2z}{dt^2}$, or which is the same thing, on $\frac{d^2.\rho s}{dt^2}$, will be sensible only in

the term $\rho \dfrac{d^2 s}{dt^2}$. Hence, equating the perturbation and its effect, we have

$$\rho \frac{d\theta}{dt}\left(\cos(\theta-\gamma)\frac{dk}{dt} + k\sin(\theta-\gamma)\frac{d\gamma}{dt}\right) + \frac{dR}{dz} = 0 \ldots (4),$$

and eliminating in turn $\dfrac{dk}{dt}$, $\dfrac{d\gamma}{dt}$, by (3), we get

$$\rho \frac{d\theta}{dt} k \frac{d\gamma}{dt} + \frac{dR}{dz}\sin(\theta-\gamma) = 0,$$

$$\rho \frac{d\theta}{dt}\frac{dk}{dt} + \frac{dR}{dz}\cos(\theta-\gamma) = 0.$$

Again, the inclination being small,

$$\frac{dR}{dk} = \frac{dR}{dz}\cdot\frac{dz}{dk} = \rho\frac{dR}{dz}\cdot\frac{ds}{dk} = \rho\frac{dR}{dz}\sin(\theta-\gamma) \text{ nearly,}$$

$$\text{and } \frac{dR}{d\gamma} = \frac{dR}{dz}\cdot\frac{dz}{d\gamma} = \rho\frac{dR}{dz}\cdot\frac{ds}{d\gamma} = -\rho k\frac{dR}{dz}\cos(\theta-\gamma);$$

$$\therefore \rho^2\frac{d\theta}{dt} k\frac{d\gamma}{dt} = -\frac{dR}{dk},$$

$$\text{and } \rho^2\frac{d\theta}{dt} k\frac{dk}{dt} = \frac{dR}{d\gamma},$$

$$\rho^2\frac{d\theta}{dt} = h = \sqrt{\mu a(1-e^2)} = \frac{\mu\sqrt{1-e^2}}{na};$$

$$\therefore \frac{d\gamma}{dt} = -\frac{1}{k}\cdot\frac{na}{\mu\sqrt{1-e^2}}\frac{dR}{dk} \ldots\ldots\ldots\ldots (5),$$

$$\frac{dk}{dt} = \frac{1}{k}\frac{na}{\mu\sqrt{1-e^2}}\frac{dR}{d\gamma} \ldots\ldots\ldots\ldots (6),$$

which agree with the known results, k being $=\tan \mathrm{I}$, or $\sin \mathrm{I}$, *quam proximè*, and γ being what Mr. Pratt denotes by Ω. (The squares, &c. of I are neglected throughout.)

R. L. E.

V.—ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

[By G. Boole.]

In an article in the first number of this Journal, (Vol. i. p. 22.) Mr. Gregory has applied the method of the separation of symbols to the Integration of Linear Differential Equations with Constant

Coefficients. The greater part of the process is at once simple and direct, which gives this method an advantage over that of the Variation of Parameters; but the reduction of the complex inverse operations to a sum of similar simple terms by means of integration by parts, is laborious and tedious, and may be very greatly abbreviated by a method which I propose here to exhibit.

If we represent the general equation

$$\left(\frac{d^n}{dx^n} + A_1 \frac{d^{n-1}}{dx^{n-1}} + A_2 \frac{d^{n-2}}{dx^{n-2}} + \&c. + A_n\right) y = X \ldots (1),$$

$$\text{by } f\left(\frac{d}{dx}\right) y = X,$$

we deduce, as is done in the paper referred to,

$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X \ldots\ldots\ldots\ldots (2).$$

Now, instead of splitting the operating factors into simple binomial factors, and operating with them in succession, which renders it necessary to simplify the result by integration by parts, we may at once resolve the general operating factors into the sum of a number of simple binomial factors, exactly as in ordinary Algebra a rational fraction is decomposed into the sum of a number of simple fractions. The expression

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1},$$

is the same in form as the rational fraction

$$\{f(z)\}^{-1} = \frac{1}{z^n + A_1 z^{n-1} + A_2 z^{n-2} + \&c. + A_n}.$$

Now the method of the resolution of this into a sum of partial fractions, is independent of any properties of the variable, except the three which have been shown by Mr. Gregory (Vol. I. p. 31.) to be common to the symbol $\frac{d}{dx}$, and to the algebraical symbols generally supposed to represent numbers. Consequently the same means which enable us to determine the form of the partial fractions in ordinary Algebra, may be applied to the circumstances of the case now under consideration. This, it will be seen, is nothing more than a farther extension of the application of the principles on which the whole method of the separation of symbols is founded. It is not necessary therefore to repeat the process of reasoning by which we arrive at the conclusion, that $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ may be expanded into a sum of partial operations, the same in form as those into which $\{f(z)\}^{-1}$ may be resolved. Still less necessary is it to work out the actual result by employing the symbol $\frac{d}{dx}$ in place

of z; it is quite sufficient to assume the form of the result as known, and to make use of our previous knowledge for the simplification of the problem. Let us therefore at once assume that $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ may be resolved into a series of simple inverse operations,

$$N_1\left(\frac{d}{dx}-a_1\right)^{-1}+N_2\left(\frac{d}{dx}-a_2\right)^{-1}+N_3\left(\frac{d}{dx}-a_3\right)^{-1}+\&c.$$

where a_1, a_2, a_3, &c. are the roots of the equation

$$f(z)=0.$$

Various methods may be employed to determine the coefficients N_1, N_2, &c., as may be seen in any work on Algebra. We shall not consider these at all, but shall content ourselves with assuming that

$$N_1=\frac{1}{(a_1-a_2)(a_1-a_3)\ldots\ldots(a_1-a_n)},$$

with similar forms for N_2, N_3, &c. These being supposed to be known, the resolution of $\left(f\frac{d}{dx}\right)^{-1}$ is complete, and there merely remains the transformation of the operation $\left(\frac{d}{dx}-a\right)^{-1}$ into the form of an integral by means of the theorem

$$\left(\frac{d}{dx}-a\right)^n=\varepsilon^{ax}\left(\frac{d}{dx}\right)^n\varepsilon^{-ax}.$$

Applying these principles, therefore, to the resolution of the equation (2), we obtain as our result

$$y=N_1\,\varepsilon^{a_1x}\int\varepsilon^{-a_1x}\,X\,dx+N_2\,\varepsilon^{a_2x}\int\varepsilon^{-a_2x}\,X\,dx+\&c.\ldots\ldots(3),$$

which is the simplest and most symmetrical form into which the solution of the equation can be brought.

If we suppose r of the roots of the equation

$$f(z)=0,$$

to be equal to each other, then the resolution into simple operations will give rise to an expression of the form

$$M\left(\frac{d}{dx}-a\right)^{-r}X+M_1\left(\frac{d}{dx}-a\right)^{-(r-1)}X+\&c.+M_{r-1}\left(\frac{d}{dx}-a\right)^{-1}X$$
$$+N_1\left(\frac{d}{dx}-a_1\right)^{-1}X+N_2\left(\frac{d}{dx}-a_2\right)^{-1}X+\&c.$$

the terms in the second line being those arising from the unequal roots. The coefficients M, M_1, &c., can easily be determined by the usual process of differentiation; the result of which would give us, if we suppose $f(z)=(x-a)^r\,\phi(z)$,

$$M_p=\frac{1}{1\,.\,2\ldots p}\left(\frac{d}{dz}\right)^p\{\phi(z)\}^{-1},\quad\text{when}\quad z=a.$$

The forms of these coefficients then being known, we have

$$y = \varepsilon^{ax} \{\mathrm{M}\textstyle\int^r \varepsilon^{-ax} \mathrm{X}\, dx^r + \mathrm{M}_1 \int^{r-1} \varepsilon^{-ax} \mathrm{X}\, dx^{r-1} + \&c.\}$$
$$+ \mathrm{N}_1 \varepsilon^{a_1 x} \textstyle\int \varepsilon^{-a_1 x} \mathrm{X}\, dx + \mathrm{N}_2 \varepsilon^{a_2 x} \int \varepsilon^{-a_2 x} \mathrm{X}\, dx + \&c. \ldots\ldots (4).$$

If the equation $f(z) = 0$ involve impossible roots, as these enter by pairs, the corresponding pairs of binomial operations may be always reduced, so that the result of the operations shall involve possible circular functions instead of impossible exponentials. Let

$$a + \beta\sqrt{-1}, \quad a - \beta\sqrt{-1},$$

be a pair of impossible roots, which for the sake of generality we shall suppose to be repeated r times in $f(z) = 0$; then the solution will consist of a series of pairs of terms, the coefficient of one term in each pair being the same function of $a + \beta\sqrt{-1}$ that the other is of $a - \beta\sqrt{-1}$. By the theory of impossible quantities these coefficients may be represented by $\mathrm{C}+\mathrm{D}\sqrt{-1}$ and $\mathrm{C}-\mathrm{D}\sqrt{-1}$, so that the series may be represented by

$$\Sigma \{(\mathrm{C} + \mathrm{D}\sqrt{-1})\, \varepsilon^{(a+\beta\sqrt{-1})x} \textstyle\int^p \varepsilon^{-(a+\beta\sqrt{-1})x} \mathrm{X}\, dx^p\}$$
$$+ \Sigma \{(\mathrm{C} - \mathrm{D}\sqrt{-1})\, \varepsilon^{(a-\beta\sqrt{-1})x} \textstyle\int^p \varepsilon^{-(a-\beta\sqrt{-1})x} \mathrm{X}\, dx^p\},$$

where the Σ refers to p, which is to receive all integer values from 1 up to r.

Now
$$\varepsilon^{\beta x\sqrt{-1}} = \cos\beta x + \sqrt{-1}\sin\beta x,$$
$$\varepsilon^{-\beta x\sqrt{-1}} = \cos\beta x - \sqrt{-1}\sin\beta x.$$

Consequently the two sums are reduced to

$$2\Sigma\,.\,\varepsilon^{ax}\left\{\begin{array}{l}(\mathrm{C}\cos\beta x - \mathrm{D}\sin\beta x)\int^p \varepsilon^{-ax}\cos\beta x\,\mathrm{X}\,dx^p \\ +(\mathrm{C}\sin\beta x + \mathrm{D}\cos\beta x)\int^p \varepsilon^{-ax}\sin\beta x\,\mathrm{X}\,dx^p\end{array}\right. \ldots (5),$$

where the index of integration receives every value from 1 up to r.

As an example of the application of this method, let us take the equation

$$\frac{d^n y}{dx^n} - y = \mathrm{X},$$

n being even.

The factors of $z^n - 1 = 0$ are all included in the formula $\cos\theta \pm \sqrt{-1}\sin\theta$, where

$$\theta = \frac{2\lambda\pi}{n},$$

λ receiving all values from 0 to $\dfrac{n}{2}$.

Therefore any pair of terms of the solution may be represented by

$$\mathrm{A}\left(\frac{d}{dx} - a\right)^{-1}\mathrm{X} + \mathrm{B}\left(\frac{d}{dx} - b\right)^{-1}\mathrm{X},$$

where $a = \cos\theta + \sqrt{-1}\sin\theta$, $b = \cos\theta - \sqrt{-1}\sin\theta$.

We easily find $A = \frac{a}{n}$, $B = \frac{b}{n}$, so that the expression (5) becomes in this case, since $p = 1$, only

$$\frac{2}{n}\left[\cos\theta\cos(x\sin\theta) - \sin\theta\sin(x\sin\theta)\right]\int\epsilon^{-x\cos\theta}\cos(x\sin\theta)\mathrm{X}dx$$

$$+\ \frac{2}{n}\left[\cos\theta\sin(x\sin\theta) + \sin\theta\cos(x\sin\theta)\right]\int\epsilon^{-x\cos\theta}\cos(x\sin\theta)\mathrm{X}dx$$

$$= \frac{2}{n}\left[\cos(x\sin\theta + \theta)\int\epsilon^{-x\cos\theta}\cos(x\sin\theta)\,\mathrm{X}dx\right.$$

$$\left. + \sin(x\sin\theta + \theta)\int\epsilon^{-x\cos\theta}\sin(x\sin\theta)\,\mathrm{X}dx\right].$$

Precisely the same mode of procedure is applicable to equations in Finite Differences. If we have an equation of the form

$$u_{x+n} + A_1 u_{x+n-1} + A_2 u_{x+n-2} + \ldots\ldots + A_n u_x = \mathrm{X},$$

and introduce (as Mr. Gregory has done, Vol. I. p. 55.) a symbol D characterized by the property

$$\mathrm{D}u_x = u_{x+1},\ \mathrm{D}u_{x+1} = u_{x+2},\ \&c.,$$

our equation assumes the form

$$(\mathrm{D}^n + A_1\mathrm{D}^{n-1} + A_2\mathrm{D}^{n-2} + \ldots\ldots + A_n)\,u_x = \mathrm{X}.$$

Whence we find

$$u_x = (\mathrm{D}^n + A_1\mathrm{D}^{n-1} + A_2\mathrm{D}^{n-2} + \ldots\ldots + A_n)^{-1}\,\mathrm{X}.$$

Then by the method of rational fractions this can be decomposed into a sum of binomial operations of the form

$$u_x = \mathrm{N}_1(\mathrm{D} - a_1)^{-1}\mathrm{X} + \mathrm{N}_2(\mathrm{D} - a_2)^{-1}\mathrm{X} + \mathrm{N}_3(\mathrm{D} - a_3)^{-1}\mathrm{X} + \ldots\ldots$$

where N_1, N_2, &c. have the same form as the similar coefficients in the solution of the differential equation, and a_1, a_2, a_3, &c. are the roots of

$$z^n + A_1 z^{n-1} + \ldots\ldots + A_n = 0.$$

Now by the theorem, Vol. i. p. 55, we have

$$(\mathrm{D} - a)^n\,\mathrm{X} = a^{x+n}\,\Delta^n(\mathrm{X}a^{-x}),$$

so that the solution becomes

$$u_x = \mathrm{N}_1 a_1^{x-1}\,\Sigma(\mathrm{X}a_1^{-x}) + \mathrm{N}_2 a_2^{x-1}\,\Sigma(\mathrm{X}a_2^{x-1}) + \ldots\ldots(6.)$$

If there be r roots equal to a, we shall obtain by the same method as that employed in the corresponding case of differential equations

$$u_x = \mathrm{M}a^{x-n}\,\Sigma^n(\mathrm{X}a^{-x}) + \mathrm{M}_1 a^{x-n+1}\,\Sigma^{n-1}(\mathrm{X}a^{-x}) +$$

$$+ \text{ terms involving the unequal roots}\ldots\ldots.(7).$$

If there be r pairs of impossible roots, we shall find, by adapting to the preceding expression the notation and reasoning previously employed, the following expression for the general term of u_x so far as it depends on the equal impossible roots

$$(\mathrm{C} + \mathrm{D}\sqrt{-1})(\alpha + \beta\sqrt{-1})^{x-p}\,\Sigma^p\{\mathrm{X}(\alpha + \beta\sqrt{-1})^{-x}\}$$

$$+ (\mathrm{C} - \mathrm{D}\sqrt{-1})(\alpha - \beta\sqrt{-1})^{x-p}\,\Sigma^p\{\mathrm{X}(\alpha - \beta\sqrt{-1})^{-x}\}.$$

To reduce this into a more convenient shape, let us assume

$$\alpha = \rho \cos \theta, \ \beta = \rho \sin \theta,$$

whence we have

$$(\alpha + \beta \sqrt{-1})^{x-p} = \rho^{x-p} \{\cos (x-p) \theta + \sqrt{-1} \sin (x-p) \theta\},$$

and similarly for the other expressions.

Substituting these values and reducing, we obtain for the general term of u_x

$$2\rho^{x-p} \left\{ \begin{array}{l} C \cos (x-p) \theta - D \sin (x-p) \theta . \Sigma^p [\rho^{-x} X \cos (x\theta)] \\ + C \sin (x-p) \theta + D \cos (x-p) \theta . \Sigma^p [\rho^{-x} X \sin (x\theta)] \end{array} \right. \ldots (8.)$$

The complete integral, so far as it depends on the pairs of impossible roots, will consist of a series of terms similar to the above, in which the index p receives every value from 1 to r.

The analogy between (3), (4), (5), and (6), (7), (8), is very remarkable, and unless we employed a method of solution common to both problems, it would not be easy to see the reason for so close a resemblance in the solution of two different kinds of equations. But the process which I have here exhibited shows, that the form of the solution depends solely on the method of decomposing the original operating factor; and this decomposition is effected by means of processes which are common to the two operations under consideration, being founded only on the common laws of the combinations of the symbols.

It is thus seen that every step of the solution of Differential Equations and equations of Finite Differences is reduced to the known theorems of ordinary Algebra, with the exception of the two theorems

$$\left(\frac{d}{dx} - a\right)^n X = \epsilon^{ax} \left(\frac{d}{dx}\right)^n \epsilon^{-ax} X, \text{ and } (D - a)^n X = a^{x+n} \Sigma^n (a^{-x} X),$$

which are necessary for passing to the interpretation of the expressions at which we arrive. This seems to be as great a simplification of the problem as the present state of mathematics admits of, for any further improvement must involve the invention of new processes for the treatment of ordinary algebraical expressions. With such we are not at present concerned; our object being to reduce the more complicated processes of the higher analysis to the simpler results which have been already obtained, and which may be looked on in the military phrase as bases for our further operations.

Waddington, near Lincoln.

VI.—INVESTIGATION OF THE ABERRATION IN RIGHT ASCENSION AND DECLINATION.

THE following investigation of the formulæ for Aberration in Right Ascension and Declination, will be found to be more simple than that given in Maddy's *Astronomy*, p. 214.

Let ΥL (fig. 4.) be the ecliptic, ΥE the equator, P its pole, T a point 90^0 behind the place of the Sun, S the place of the star; then ST will be the plane of aberration.

Let $\Upsilon N = a$, $PN = \delta$, Sun's longitude $= \odot$, and $L\Upsilon E = \omega$.

1st. For the Aberration in Declination: produce SN to a point Q, such that $SQ = 90^0$, and join TQ, TN. If A be the coefficient of aberration, and $\Delta\delta$ the aberration in declination,

$$\Delta\delta = - \text{A} \sin \text{ST} . \cos \text{TSQ} = - \text{A} \cos \text{TQ},$$

as TSQ is a quadrantal triangle.

$$\text{But } \cos \text{QT} = \cos \text{TN} . \cos \text{QN} + \sin \text{TN} . \sin \text{QN} . \sin \text{TN}\Upsilon,$$

$$\text{and } \cos \text{TN} = \cos \text{T}\Upsilon . \cos \Upsilon\text{N} + \sin \text{T}\Upsilon . \sin \Upsilon\text{N} . \cos \text{T}\Upsilon\text{N},$$

$$= \sin \odot \cos a - \cos \odot \sin a \cos \omega.$$

$$\text{Also, } \sin \text{TN} . \sin \text{TN}\Upsilon = \sin \text{T}\Upsilon . \sin \text{T}\Upsilon\text{N} = \cos \odot \sin \omega.$$

Substituting these values, and putting $90^0 - \delta$ for QN, we find

$$\Delta\delta = - \text{A} \{\sin \delta (\sin \odot \cos a - \cos \odot \sin a \cos \omega) + \cos \delta \cos \odot \sin \omega\}.$$

2nd. For the Aberration in Right Ascension: produce $N\Upsilon$ to a point R, such that $NR = 90^0$, and join RS, RT. Then, if Δa be the aberration in right ascension,

$$\Delta a = - \frac{\text{A}}{\cos \delta} \sin \text{ST} . \cos \text{TSR} = - \frac{\text{A}}{\cos \delta} \cos \text{RT}.$$

$$\text{But } \cos \text{RT} = \cos \text{T}\Upsilon . \cos \text{R}\Upsilon + \sin \text{T}\Upsilon . \sin \text{R}\Upsilon \cos \text{R}\Upsilon\text{T},$$

$$= \sin \odot \sin a + \cos a \sin \odot \cos \omega.$$

Consequently,

$$\Delta a = - \frac{\text{A}}{\cos \delta} \{\sin \odot \sin a + \cos a \sin \odot \cos \omega\}.$$

R. L. E.

VII.—ON THE SYMPATHY OF PENDULUMS.

BY the Sympathy of Pendulums is meant the effect on the motions of different pendulums produced by their mutual action, when their points of suspension have any elastic or moveable connexion.

The phenomenon is a striking one, and presents itself in a marked manner to those who are engaged in the art of clock-making. It has been observed by them, that if the pendulums of two clocks, the times of the oscillation of which are different, be so situated that the motion of the one can be in any way communicated to the other—as, for instance, by their centres of suspension being attached to the same beam—the motions of the two pendulums are entirely altered by their mutual action, the periods of both tending to become the same, and the extent of oscillation continually changing. This becomes a serious practical inconvenience, and it is necessary to take precautions to prevent the influence of the one pendulum being communicated to the other. Daniel Bernoulli appears to have been the first who took notice of this phenomenon, at least with any reference to theory; but the case which attracted his attention was far more simple than that to which we have alluded. It was that of the motion of the two scales of a balance, when one has had an oscillatory motion communicated to it. The following is his narration of the phenomenon as he observed it. (*Nova Commen. Petrop.*, Vol. XIX. p. 281.)

"Cum aliquando in libra, majori eaque subpigra, alteram lancem "forte fortuna ad latus diducerem, moxque rursus dimitterem, "accidit utique ut protinus hinc inde oscillaret nec ab initio lanx "opposita de loco moveretur: mox autem et hæc quoque agitari "sensimque majores oscillationes formare, dum e contrario lanx "prior motum suum oscillatorium gradatim perderet tandemque "fere quiesceret; hoc ipso momento altera maximum motionis "gradum, initiali lancis sociæ fere æqualem, attingebat: tunc "ordine contrario eædem mutationes repetebantur, usque dum "prima lanx motum suum primitivum integrum resumeret so- "ciaque quieti ad momentum redderetur; hæc autem oscillationum "communicatio ac reciprocatio diu satis sese manifestabat."

Bernoulli does not seem to have attempted a direct solution of the dynamical problem which this experiment suggested, but contents himself with adducing it as an instance in support of his principle of the coexistence of small oscillations. In the same volume of the *Petersburgh Memoirs,* however, Euler in two papers considers the question theoretically. It is clear that the experiment of Bernoulli admits of being performed in two different ways: the original displacement of the scale may either be in the vertical plane passing through the beam of the balance, or it may be in any other plane. Euler only considers the first case, though another one—that when the original displacement is perpendicular to the vertical plane passing through the beam—is also interesting, and admits of as easy a solution. In his first memoir, Euler supposes that the centre of suspension of the balance is in the same line as the centres of suspension of the two scales; and on investigating the result on this supposition, he finds that the interchange of motion described by Bernoulli could not take place, though the motion of the scale originally put in motion would be different from that

which it would have if suspended from a fixed point of support. This result is confirmed by the simple consideration, that when the centre of suspension of the beam is in the line joining the centres of suspension of the scales, these will only receive vertical motions from the motion of the beam; and consequently, if we suppose the second scale to be originally at rest, it will have no horizontal motion communicated to it, so as to cause it to oscillate in a horizontal direction.

In the other memoir Euler considers the case of the centre of suspension being above the line joining the points of suspension, as indeed would be the case in an ordinary balance which is usually suspended by some higher point. This investigation we shall here give, adhering pretty closely to the process adopted by Euler, as it is always both interesting and instructive to see the mode in which the first writers attacked such a problem as this.

Let O (fig. 5.) be the point of suspension of the whole balance, G its centre of gravity, AB the beam, P and Q the scales, which are here supposed to be material points. Draw aOb horizontal, $A\alpha$, $B\beta$ vertical. Let $AC = CB = a$, $OC = b$, $OG = c$, $AP = BQ = l$, $Mk^2 =$ moment of inertia of the beam, m the mass of P and of Q supposed to be equal.

Let φ be the angle which, at the time t, the beam makes with the horizon.

$$\text{Let}\quad PA\alpha = \eta, \quad QB\beta = \theta,$$

$$Op = x, \quad Pp = y, \quad Oq = x', \quad Qq = y'.$$

Then
$$x = a\cos\phi + b\sin\phi + l\sin\eta,$$
$$y = b\cos\phi - a\sin\phi + l\cos\eta,$$
$$x' = a\cos\phi - b\sin\phi - l\sin\theta,$$
$$y' = a\sin\phi + b\cos\phi + l\cos\theta.$$

Let P, Q, be the tensions of AP and BQ.

The equations of their motions are

$$(1)\quad \frac{d^2x}{dt^2} = -\frac{Pg\sin\eta}{m}. \qquad (2)\quad \frac{d^2y}{dt^2} = g - \frac{Pg\cos\eta}{m}.$$

$$(3)\quad \frac{d^2x'}{dt^2} = \frac{Qg\sin\theta}{m}. \qquad (4)\quad \frac{d^2y'}{dt^2} = g - \frac{Qg\cos\theta}{m}.$$

For the motion of the beam we have its own weight at G tending to turn it back to its original position, and the tensions of the strings acting in different directions. The moment of the couple arising from the weight of the beam is

$$Mgc\cos GOA = Mgc\sin\phi.$$

The moment of the couple arising from the tension P is

$$Pg\,Om = Pg\,\{a\cos(\eta - \phi) - b\sin(\eta - \phi)\}.$$

Both these are negative, as tending to bring back the beam to its original position. The moment of the couple arising from Q is

$$Qg\,\{a\cos(\phi - \theta) - b\sin(\varphi - \theta)\}.$$

Consequently we have, for the motion of the beam, the equation

$$(5)\quad \frac{d^2\phi}{dt^2} = -\frac{g}{Mk^2}\left\{\begin{array}{l} Mc \sin\phi + P\,[a\cos(\eta-\phi) - b\sin(\eta-\phi)] \\ \qquad - Q\,[a\cos(\phi-\theta) - b\sin(\phi-\theta)] \end{array}\right\}$$

These five simultaneous equations, if solved, would serve to determine all the circumstances of the motion, but under their present form the solution is impracticable. To render it possible, we must suppose the displacements to be very small, so that we may put the arc for its sine and unity for the cosine; by this means we find

$$x = a + b\phi + l\eta, \qquad y = b - a\phi + l,$$
$$x' = a - b\phi - l\theta, \qquad y' = a\phi + b + l.$$

Also, the tensions of the strings may be supposed not to be changed, but to remain equal to the weight. Making these substitutions in the five equations of motion, the second and fourth disappear, and there remain

$$b\frac{d^2\phi}{dt^2} + l\frac{d^2\eta}{dt^2} = -g\eta,$$

$$b\frac{d^2\phi}{dt^2} + l\frac{d^2\theta}{dt^2} = -g\theta,$$

$$Mk^2\frac{d^2\phi}{dt^2} = -g\,\{(Mc + 2mb)\,\phi - mb\,(\eta + \theta)\},$$

and by means of these three simultaneous equations we can easily determine ϕ, η, θ.

Let $\frac{g}{l} = n^2, \quad \frac{b}{l} = h, \quad \frac{Mc + 2mb}{Mk^2}\,g = p^2, \quad -\frac{mbg}{Mk^2} = q.$

Then the equations may be put under the form

$$(1)\quad h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\eta = 0.$$

$$(2)\quad h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\theta = 0.$$

$$(3)\quad \left(\frac{d^2}{dt^2} + p^2\right)\phi + q\,(\eta + \theta) = 0.$$

*Add together (1) and (2), which gives

$$(4)\quad 2h\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)(\eta + \theta) = 0.$$

Subtract (2) from (1), which gives

$$(5)\quad \left(\frac{d^2}{dt^2} + n^2\right)(\eta - \theta) = 0.$$

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* For this method of integrating simultaneous differential equations, see *Mathematical Journal*, Vol. I., p. 173.

Operate on (3) with $\left(\frac{d^2}{dt^2} + n^2\right)$, multiply (4) by q, and subtract it from the former. Then

$$\left(\frac{d^2}{dt^2} + n^2\right)\left(\frac{d^2}{dt^2} + p^2\right)\phi - 2hq\frac{d^2\phi}{dt^2} = 0.$$

If $-\mu_1^2$, $-\mu_2^2$, be the roots of

$$z^2 + (n^2 + p^2 - 2hq)\,z + n^2p^2 = 0,$$

this may be put under the form

$$(6)\quad \left(\frac{d^2}{dt^2} + \mu_1^2\right)\left(\frac{d^2}{dt^2} + \mu_2^2\right)\phi = 0.$$

The integral of which is

(A) $\phi = C_1 \cos(\mu_1 t + a_1) + C_2 \cos(\mu_2 t + a_2)$.

Substituting this value of ϕ in (3), we find

$$\eta + \theta = \frac{C_1}{q(\mu_1^2 - p^2)}\cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)}\cos(\mu_2 t + a_2).$$

Integrating (5), we have

$$\eta - \theta = C\cos(nt + a);$$

whence

$$(B)\quad 2\eta = \frac{C_1}{q(\mu_1^2 - p^2)}\cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)}\cos(\mu_2 t + a_2) + C\cos(nt + a),$$

$$(C)\quad 2\theta = \frac{C_1}{q(\mu_1^2 - p^2)}\cos(\mu_1 t + a_1) + \frac{C_2}{q(\mu_2^2 - p^2)}\cos(\mu_2 t + a_2) - C\cos(nt + a).$$

The equations (A), (B), (C), are the complete solution of the problem, and involve six arbitrary constants, viz. C, C_1, C_2, a, a_1, a_2, which may be determined so as to suit the original circumstances, and according to the nature of these the solution will assume different shapes. To suit the experiment of Bernoulli, we must suppose, when $t = 0$, that

$$\phi = 0, \qquad \eta = \varepsilon, \qquad \theta = 0,$$

$$\frac{d\phi}{dt} = 0, \qquad \frac{d\eta}{dt} = 0, \qquad \frac{d\theta}{dt} = 0.$$

The last three conditions give us $a = a_1 = a_2 = 0$; and these values substituted in the others reduce them to

$$0 = C_1 + C_2, \qquad \varepsilon = C,$$

$$q\varepsilon = \frac{C_1}{\mu_1^2 - p^2} + \frac{C_2}{\mu_2^2 - p^2};$$

whence we find

$$C_1 = q\varepsilon\,\frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_2^2 - \mu_1^2}, \qquad C_2 = q\varepsilon\,\frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_1^2 - \mu_2^2}.$$

Substituting these values the equations become

$$\phi = q\,\frac{\varepsilon(\mu_1^2-p^2)(\mu_2^2-p^2)}{\mu_2^2-\mu_1^2}(\cos\mu_1 t-\cos\mu_2 t),$$

$$2\eta = \frac{\varepsilon}{\mu_2^2-\mu_1^2}\{(\mu_2^2-p^2)\cos\mu_1 t-(\mu_1^2-p^2)\cos\mu_2 t\}+\varepsilon\cos nt,$$

$$2\theta = \frac{\varepsilon}{\mu_2^2-\mu_1^2}\{(\mu_2^2-p^2)\cos\mu_1 t-(\mu_1^2-p^2)\cos\mu_2 t\}-\varepsilon\cos nt.$$

Observing that $\mu_1^2+\mu_2^2 = n^2+p^2-2hq$, and $\mu_1^2\mu_2^2 = n^2p^2$, the values of C_1 and C_2 are reduced to

$$C = \frac{2\varepsilon hp^2q^2}{\mu_2^2-\mu_1^2}, \qquad C_2 = \frac{2\varepsilon hp^2q^2}{\mu_1^2-\mu_2^2};$$

so that we find

$$\phi = \frac{2\varepsilon hp^2q^2}{\mu_2^2-\mu_1^2}(\cos\mu_1 t-\cos\mu_2 t),$$

$$2\eta = \frac{2\varepsilon hp^2q}{\mu_2^2-\mu_1^2}\left(\frac{\cos\mu_1 t}{\mu_1^2-p^2}-\frac{\cos\mu_2 t}{\mu_2^2-p^2}\right)+\varepsilon\cos nt,$$

$$2\theta = \frac{2\varepsilon hp^2q}{\mu_2^2-\mu_1^2}\left(\frac{\cos\mu_1 t}{\mu_1^2-p^2}-\frac{\cos\mu_2 t}{\mu_2^2-p^2}\right)-\varepsilon\cos nt.$$

It appears from these expressions, that the motion of the beam is compounded of two oscillations of different periods, $\frac{2\pi}{\mu_1}$ and $\frac{2\pi}{\mu_2}$. The relations which these oscillations bear to those which the beam and the scales would separately make, may be easily shown. By means of equations (3) and (4), we see that μ_1^2 and μ_2^2 must satisfy the equation

$$(\mu^2-n^2)(\mu^2-p^2)+2hq\mu^2 = 0.$$

If the mass of the beam be very large, we may suppose p^2 to be less than n^2, and the equation for μ^2 may be put under the forms

$$\mu_1^2 = n^2-\frac{2hq\mu_1^2}{\mu_1^2-p^2},$$

$$\text{and}\quad \mu_2^2 = p^2+\frac{2hq\mu_2^2}{n^2-\mu_2^2},$$

which show that μ_1^2 is less than n^2, and μ_2^2 greater than p^2.

Hence it appears that the period of the one part of the oscillation of the beam is greater than the period of the natural oscillation of the scales, while the period of the other oscillation is less than that of the beam and scales considered as one mass. The motion of the scales consists of these same oscillations, with the addition of one, the period of which is that of a pendulum of the same length as the suspending string.

If we suppose the vibrations of the scales to take place in a plane perpendicular to a vertical plane passing through the beam,

the expressions become somewhat simpler. We shall not, in this case, go so much into detail as in the last, but shall at once suppose the displacements to be so small, that the forces of restitution may be considered as proportional to them.

Let AB (fig. 6.) be the original position of the beam, PQ its position at the time t; p, q the projections of the positions of scales considered as material points at the same time.

Let $AC = BC = a$, $AP = BQ = z$, $Pp = x$, $Qq = y$, Mk^2 be the moment of inertia of the beam round C, m the mass of each weight, l the length of the string by which each weight is suspended. Then the equations of motion will be

$$\frac{d^2(x+z)}{dt^2} + \frac{g}{l}x = 0 \text{ .... } (1),$$

$$\frac{d^2(y+z)}{dt^2} + \frac{g}{l}y = 0 \text{ .... } (2),$$

$$\frac{d^2z}{dt^2} - \frac{g}{l}\frac{ma^2}{Mk^2}(x+y) = 0 \text{ .... } (3).$$

Subtracting (2) from (1), we have

$$\frac{d^2(x-y)}{dt^2} + \frac{g}{l}(x-y) = 0.$$

Add (1) and (2), and subtract (3) multiplied by 2; then

$$\frac{d^2(x+y)}{dt^2} + \frac{g}{l}\left(1 + 2\frac{ma^2}{Mk^2}\right)(x+y) = 0.$$

Let $\frac{g}{l} = n^2$, $\frac{g}{l}\left(1+2\frac{ma^2}{Mk^2}\right) = n'^2$; then these equations become

$$\frac{d^2(x-y)}{dt^2} + n^2(x-y) = 0,$$

$$\frac{d^2(x+y)}{dt^2} + n'^2(x+y) = 0.$$

Integrating, we have

$$x - y = C\cos(nt + \alpha),$$
$$x + y = C_1\cos(n't + \alpha_1).$$

If we suppose that at the beginning of the motion

$$x = c, \qquad y = 0, \qquad \frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0,$$

these equations become

$$x - y = C\cos nt,$$
$$x + y = C\cos n't;$$

$$\text{whence} \quad x = C\cos\frac{n'-n}{2}t\cos\frac{n'+n}{2}t,$$
$$y = -C\sin\frac{n'-n}{2}t\sin\frac{n'+n}{2}t.$$

Substituting the value of $x + y$ which has been found in (3), and integrating, we find

$$z = -\frac{g}{l}\frac{ma^2}{Mk^2}\frac{c}{n'^2}\cos n't + At + B.$$

If at the beginning of the motion we suppose $z = 0$, $\frac{dz}{dt} = 0$,

we shall find $A = 0$, $B = \frac{g}{l}\frac{ma^2}{Mk^2}\frac{c}{n'^2}$, so that

$$z = \frac{g}{l}\frac{ma^2}{Mk^2}\frac{c}{n'^2}(1 - \cos n't),$$

$$\text{or} \quad z = \frac{2g}{l}\frac{ma^2}{Mk^2}\frac{c}{n'^2}\sin^2\frac{n't}{2}.$$

If we suppose the beam to have an original angular velocity given to it, then A will not be 0, and the expression for z will no longer be simply periodic, but will increase continually with the time. This will also appear from the consideration, that there is no force independent of the oscillations of the scale acting on the beam, so that any originally impressed velocity will not be destroyed, but will continue to carry the beam round with a motion subject to periodic inequalities.

If we suppose the mass of the beam to be very great in comparison with the masses of the weights, so that $\frac{ma^2}{Mk^2}$ is very small, n is very nearly equal to n', $\frac{n'-n}{2}$ is very small, and $\sin\frac{n'-n}{2}t$ and $\cos\frac{n'-n}{2}t$ vary very slowly.

So that we may represent our result as that of two pendulums, whose arcs of vibration are respectively

$$C\cos\frac{n'-n}{2}t \text{ and } C\sin\frac{n'-n}{2}t.$$

These are complementary; they show that the arc of vibration of the first pendulum will gradually diminish, and that of the second increase, till after a time $= \frac{2}{n'-n}\cdot\frac{\pi}{2}$ they have interchanged motions and the converse process is repeated, and the system returns to its original state after a time $= \frac{2\pi}{n'-n}$.

The common time of oscillation is that of a pendulum whose length is

$$g\cdot\frac{4}{(n+n')^2} = l\left(1 - \frac{ma^2}{Mk^2}\right) \text{ nearly.}$$

The most general case of the influence of one pendulum on another, when the motions as we have supposed are all in the same horizontal direction and infinitesimal, will be when, calling A and B the points of support, each of these when disturbed performs vibrations in known times; and a disturbance given to A communicates a known motion to B, and *vice versâ.*

To investigate the motion in this case, let u, v be the coordinates of A and B, $u+x$, $v+y$ of the balls suspended to them; then we have the equations

$$\frac{d^2(u+x)}{dt^2} + m^2x = 0,$$

$$\frac{d^2(v+y)}{dt^2} + n^2y = 0,$$

$$\frac{d^2u}{dt^2} + p^2u - ax - fv = 0,$$

$$\frac{d^2v}{dt^2} + q^2v - by - gu = 0.$$

A solution of these equations is

$$x = \text{RM} \cos(\sqrt{\rho}\, t - r), \qquad u = \text{RP} \cos(\sqrt{\rho}\, t - r),$$

$$y = \text{RN} \cos(\sqrt{\rho}\, t - r), \qquad v = \text{RQ} \cos(\sqrt{\rho}\, t - r);$$

and to determine ρ we get the equation

$$\{(\rho - m^2)(\rho - p^2) - a\rho\}\{(\rho - n^2)(\rho - q^2) - b\rho\} - fg(\rho - m^2)(\rho - n^2) = 0.$$

The four values of ρ determined from this equation are all positive, and therefore the angular functions real. The complete solution is the sum of the particular solutions, therefore

$$x = \text{R}_1\text{M}_1 \cos(\sqrt{\rho_1}\, t - r_1) + \text{R}_2\text{M}_2 \cos(\sqrt{\rho_2}\, t - r_2) + \text{R}_3\text{M}_3 \cos(\sqrt{\rho_3}\, t - r_3) + \text{R}_4\text{M}_4 \cos(\sqrt{\rho_4}\, t - r_4),$$

$$y = \text{R}_1\text{N}_1 \cos(\sqrt{\rho_1}\, t - r_1) + \text{R}_2\text{N}_2 \cos(\sqrt{\rho_2}\, t - r_2) + \text{R}_3\text{N}_3 \cos(\sqrt{\rho_3}\, t - r_3) + \text{R}_4\text{N}_4 \cos(\sqrt{\rho_4}\, t - r_4).$$

If three of the quantities R are equal to zero, the vibrations of the two pendulums are isochronous, and there are therefore four modes of this isochronous vibration. In all cases the pendulums affect each other, so that none of the points oscillates in its natural time. The extreme generality of the equations we have assumed, and consequently of the solution derived from them, prevents us from interpreting our result in a more precise manner. For this purpose it would be necessary to assign some relations between the constants in the equations, but it would lead us too far if we were to attempt any such investigation; and we may add, that any particular case will in general be more easily solved by a direct reference to its own circumstances than by a reduction of the general solution.

D. G. S.

VIII.—ON THE EXPANSION OF COSINES AND SINES OF MULTIPLE ARCS IN ASCENDING POWERS OF THE COSINES AND SINES OF THE SIMPLE ARCS.

THE method adopted by Lagrange for expressing the cosine and sine of multiple arcs in terms of the powers of the cosines and sines of the simple arcs, depended on the expansion of functions of the form

$$(x+\sqrt{x^2-1})^n \text{ and } (x-\sqrt{x^2-1})^n.$$

The same method is pursued by Poinsot in his *Recherches sur l'Analyse des Sections Angulaires,* where the complete theory of these circular functions was first given; but he has also indicated another way, which is far less tedious and complicated—that of assuming the form of the series, and determining the coefficients by differentiation. As this may be useful to those who are studying the subject, we shall here briefly fill up the outline which Poinsot has sketched.

1. To expand $\cos n\theta$ in terms of $\cos\theta$ and its powers.

Assume

$$\cos n\theta = a_0 + a_1\cos\theta + \dots + a_p(\cos\theta)^p + \dots + a_{p+2}(\cos\theta)^{p+2} + \dots$$

Differentiating,

$$n\sin n\theta = \{a_1 + 2a_2\cos\theta + \dots + pa_p(\cos\theta)^{p-1} + \dots \\ + (p+2)\,a_{p+2}(\cos\theta)^{p+1} + \dots\}\sin\theta.$$

Differentiating again,

$$n^2\cos n\theta = a_1\cos\theta + \dots\dots + pa_p(\cos\theta)^p + \dots\dots \\ + (p+2)\,a_{p+2}(\cos\theta)^{p+2} + \dots\dots \\ - \{2a_2 + \dots\dots + p(p-1)\,a_p(\cos\theta)^{p-2} + \dots\dots \\ + (p+1)(p+2)\,a_{p+2}(\cos\theta)^p \dots\dots\}\sin^2\theta.$$

Putting $1-\cos^2\theta$ for $\sin^2\theta$, and taking the coefficient of $(\cos\theta)^p$, we find it to be

$$pa_p + p(p-1)\,a_p - (p+1)(p+2)\,a_{p+2},$$

and this must be equal to the coefficient of $(\cos\theta)^p$ in the first equation multiplied by n^2. Therefore, we have

$$n^2 a_p = pa_p + p(p-1)\,a_p - (p+1)(p+2)\,a_{p+2};$$

$$\text{whence } a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

By this means any coefficient is found in terms of that two places below it. Consequently the first and second coefficients are left to be determined by other means. For this purpose let $\theta = (2r+1)\,\dfrac{\pi}{2}$

in the first equation, r being any integer. Every term on the second side vanishes except the first, and we find

$$a_0 = \cos n\,(2r+1)\,\frac{\pi}{2}.$$

To find a_1, make $\theta = (2r+1)\,\frac{\pi}{2}$ in the second equation, when we obtain

$$a_1 = n\,\frac{\sin n\,(2r+1)\,\frac{\pi}{2}}{\sin\,(2r+1)\,\frac{\pi}{2}} = n\cos\,(n-1)\,(2r+1)\,\frac{\pi}{2}.$$

Starting from these values, and giving p successively all the integer values from 0 upwards, and separating the terms involving odd powers of $\cos\theta$ from those involving even powers, we find

$$\cos n\theta = \cos n\,(2r+1)\,\frac{\pi}{2}\left(1 - \frac{n^2}{1.2}(\cos\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\cos\theta)^4 - \&c.\right)$$
$$+ n\cos\,(n-1)\,(2r+1)\,\frac{\pi}{2}\left(\cos\theta - \frac{n^2-1^2}{1.2.3}(\cos\theta)^3 + \&c.\right)$$

When n is an even integer, the second line being multiplied by the cosine of an odd multiple of $\frac{\pi}{2}$ vanishes, and the first line only remains; when n is an odd integer, the first line vanishes, and the second line only remains. When n is a fraction, both lines must be retained, except for particular values of r, which cause the factor of one or other series to vanish.

2. If we assume

$$\sin n\theta = a_0 + a_1\sin\theta + a_2(\sin\theta)^2 + \&c. + a_p(\sin\theta)^p + \&c.$$

we shall obtain, by the same means as in the previous case, the same equation for determining a_{p+2}, viz.

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the above equation and its differential. We thus obtain

$$a_0 = \sin nr\pi,$$
$$a_1 = n\,\frac{\cos nr\pi}{\cos r\pi} = n\cos\,(n-1)\,r\pi\,;$$

so that, dividing the series into two parts containing the odd and the even powers, we have

$$\sin n\theta = \sin nr\pi\left(1 - \frac{n^2}{1.2}(\sin\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin\theta)^4 - \&c.\right)$$
$$+ n\cos\,(n-1)\,r\pi\left(\sin\theta - \frac{n^2-1^2}{1.2.3}(\sin\theta)^3 + \&c.\right).$$

When n is an integer the first series always vanishes, and the second is positive or negative according as $(n-1)r$ is even or odd. When n is odd the second series terminates; when n is even it goes on to infinity. When n is a fraction, both series are to be retained.

3. If we assume

$$\cos n\theta = a_0 + a_1 \sin\theta + a_2(\sin\theta)^2 + \&c. + a_p(\sin\theta)^p - \&c.$$

we obtain as before for determining a_{p+2}, the equation

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the equation and its differential. Then

$$a_0 = \cos nr\pi,$$

$$a_1 = -\frac{n\sin nr\pi}{\cos r\pi} = -n\sin(n-1)r\pi;$$

so that

$$\cos n\theta = \cos nr\pi\left(1 - \frac{n^2}{1.2}(\sin\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin\theta)^4 \quad \&c.\right)$$
$$-n\sin(n-1)r\pi\left(\sin\theta - \frac{n^2-1^2}{1.2.3}(\sin\theta)^3 + \&c.\right)$$

When n is an integer the second series always disappears, and the first series terminates when n is even, and does not terminate when n is odd. When n is a fraction, both series are retained.

4. If we assume

$$\sin n\theta = a_0 + a_1\cos\theta + \&c. + a_p(\cos\theta)^p + \&c.$$

we find as before as the condition for determining the coefficients,

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

Make $\theta = (2r+1)\frac{\pi}{2}$ in the equation and its differential. Then

$$a_0 = \sin n(2r+1)\frac{\pi}{2}.$$

$$a_1 = -n\,\frac{\cos n(2r+1)\frac{\pi}{2}}{\sin(2r+1)\frac{\pi}{2}} = n\sin(n-1)(2r+1)\frac{\pi}{2};$$

whence we find

$$\sin n\theta =$$

$$\sin n(2r+1)\frac{\pi}{2}\left(1 - \frac{n^2}{1.2}(\cos\theta)^2 + n^2\frac{(n^2-2^2)}{1.2.3.4}(\cos\theta)^2 + \&c.\right)$$
$$+n\sin(n-1)(2r+1)\frac{\pi}{2}\left(\cos\theta - \frac{n^2-1^2}{1.2.3}(\cos\theta)^3 + \&c.\right).$$

When n is an odd integer the first line, when n is even the second line only remains; but when n is fractional both series must be retained. In no case do the series ever terminate.

These four results may be put under a convenient mnemonic form if we make use of a particular notation, by which the laws of these series are assimilated to those of sines and cosines.

Let P_n be such an operation performed on n that

$$P_n^{\,0}=1, \qquad P_n^{\,1}=n,$$

$$P_n^{\,2}=(n-0)\,(n+0)=n^2, \qquad P_n^{\,3}=n(n-1)(n+1)=n(n^2-1^2),$$

$$P_n^{\,4}=n^2(n-2)(n+2)=n^2(n^2-2^2), \quad P_n^{\,5}=n\,(n^2-1^2)\,(n^2-3^2),$$

&c. &c.

from which the law of formation is evident. Then the series of (1) may be put under the form

$$\cos n\theta=\cos n\,(2r+1)\frac{\pi}{2}\left(1-\frac{P_n^{\,2}}{1.2}(\cos\theta)^2+\frac{P_n^{\,4}}{1.2.3.4}(\cos\theta)^4-\&c.\right)$$

$$+\cos\,(n-1)\,(2r+1)\frac{\pi}{2}\left(P_n^{\,1}\cos\theta-\frac{P_n^{\,3}}{1.2.3}(\cos\theta)^3+\&c.\right);$$

in which it will be seen that the series in the first line follows the law of the cosine of ($P_n\cos\theta$), and the series in the second line that of the sine of ($P_n\cos\theta$), so that the expression may be put under the form

$$\cos n\theta=\cos n\,(2r+1)\frac{\pi}{2}\cos\,(P_n\cos\theta)$$

$$+\cos\,(n-1)\,(2r+1)\frac{\pi}{2}\sin\,(P_n\cos\theta)$$

$$=\cos n\,(2r+1)\frac{\pi}{2}\cos\,(P_n\cos\theta)$$

$$\pm\sin n\,(2r+1)\frac{\pi}{2}\sin\,(P_n\cos\theta);$$

whence

$$\text{(I).}\quad \cos n\theta=\cos\left\{n\,(2r+1)\frac{\pi}{2}\mp(P_n\cos\theta)\right\}.$$

By using the same notation with respect to (2), we have

$$\sin n\theta=\sin nr\pi\left(1-\frac{P_n^{\,2}}{1.2}(\sin\theta)^2+\frac{P_n^{\,4}}{1.2.3.4}(\sin\theta)^4-\&c.\right)$$

$$+\cos\,(n-1)\,r\pi\left(P_n\sin\theta-\frac{P_n^{\,3}}{1.2.3}(\sin\theta)^3+\&c.\right)$$

$$=\sin nr\pi\cos\,(P_n\sin\theta)\pm\cos nr\pi\sin\,(P_n\sin\theta);$$

and therefore

$$\text{(II).}\quad \sin n\theta=\sin\left\{nr\pi\pm(P_n\sin\theta)\right\}.$$

Proceeding in the same way, we shall find

$$\cos n\theta = \cos nr\pi \cos (P_n \sin \theta) \mp \sin nr\pi \sin (P_n \sin \theta),$$

or

$$\text{(III).} \quad \cos n\theta = \cos \{nr\pi \pm (P_n \sin \theta)\}.$$

Similarly

$$\text{(IV).} \quad \sin n\theta = \sin \left\{ n(2r+1)\frac{\pi}{2} \mp (P_n \cos \theta) \right\}.$$

G.

IX.—ON THE LINES OF CURVATURE ON AN ELLIPSOID.

By R. L. Ellis, B.A., Trinity College.

The following investigation of the Lines of Curvature on an Ellipsoid, has the advantages of symmetry and of giving a distinct geometrical conception. The artifice on which it depends may, it is thought, be found useful on other occasions.

The symmetrical equation to the lines of curvature is

$$(b^2-c^2)\, x\, dy\, dz + (c^2-a^2)\, y\, dz\, dx + (a^2-b^2)\, z\, dx\, dy = 0 \ldots (1),$$

(see *Mathematical Journal*, Vol. I., p. 142.), where xyz are connected by the equation to the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots (2).$$

$$\text{Put} \quad \frac{x^2}{a^2} = u, \quad \frac{y^2}{b^2} = v, \quad \frac{z^2}{c^2} = w \ldots\ldots\ldots\ldots\ldots (A).$$

$$\text{Then} \quad x\, dy\, dz = \tfrac{1}{4} a \sqrt{u} \,.\, b \frac{dv}{\sqrt{v}}\, c \frac{dw}{\sqrt{w}} = \frac{abc}{4\sqrt{uvw}}\, u\, dv\, dw.$$

Hence, after the substitution and multiplying by $\dfrac{4\sqrt{uvw}}{abc}$, (1) becomes

$$(b^2-c^2)\, u\, dv\, dw + (c^2-a^2)\, v\, dw\, du + (a^2-b^2)\, w\, du\, dv = 0 \ldots (3),$$

with the relation

$$u + v + w = 1 \ldots\ldots\ldots\ldots (4).$$

Differentiate (3); then, since

$$b^2 - c^2 + c^2 - a^2 + a^2 - b^2 = 0,$$

we get

$$(b^2 - c^2)\, ud\,(dv\, dw) + (c^2 - a^2)\, vd\,(dw\, du)$$
$$+ (a^2 - b^2)\, wd\,(du\, dv) =$$

Now this is satisfied by the assumptions

$$dv\,dw = \frac{1}{f}, \quad dw\,du = \frac{1}{g}, \quad du\,dv = \frac{1}{h} \;\ldots\ldots\ldots (\mathrm{B}),$$

f, g, h, being constants.

But from (4) we deduce

$$du + dv + dw = 0 \;\ldots\ldots\ldots\; (6),$$

and (B) gives

$$du = f\,du\,dv\,dw, \quad dv = g\,du\,dv\,dw, \quad dw = h\,du\,dv\,dw.$$

$$\text{Hence} \qquad f + g + h = 0 \;\ldots\ldots\ldots\; (7),$$

which establishes a relation among the otherwise arbitrary constants f, g, h.

Now (B) implies the existence of two linear equations in u, v, w. Hence, a *particular solution* of (1) is two linear equations connecting the three variables. But the given equation (4) is linear; hence the solution in question is the one congruent to the problem.

To find the other relation in u, v, w, eliminate the differentials from (3) by means of (B), and there is

$$(b^2 - c^2)\frac{u}{f} + (c^2 - a^2)\frac{v}{g} + (a^2 - b^2)\frac{w}{h} = 0 \;\ldots\ldots\; (8).$$

Equations (4) and (5), with the relation (7), contain the complete solution of the problem. It is obvious that the apparent want of homogeneity of (B) is wholly immaterial.

Keeping in mind the values of u, v, w, given by (A), we see that the geometrical interpretation of (8) is, every line of curvature on an ellipsoid lies on a conical surface of the second order, of which the vertex is the centre of the ellipsoid.

To determine the constants, let the line of curvature pass through a point, for which the values of u, v, w, are u_1, v_1, w_1, we have

$$(b^2 - c^2)\frac{u_1}{f} + (c^2 - a^2)\frac{v_1}{g} + (a^2 - b^2)\frac{w_1}{h} = 0,$$

$$f + g + h = 0.$$

Hence, after a slight reduction,

$$(b^2 - c^2)\,u_1\frac{g}{f} + (c^2 - a^2)\,v_1\frac{f}{g} - (a^2 - b^2)\,w_1$$
$$+ (b^2 - c^2)\,u_1 + (c^2 - a^2)\,v_1 = 0 \;\ldots\ldots\; (9),$$

a quadratic in $\frac{g}{f}$, of which the roots are real and of unlike signs. This is obvious, for u_1, v_1, are essentially positive, and a, b, c, being in order of magnitude, the signs of $b^2 - c^2$ and $c^2 - a^2$ are opposite. Similarly, $\frac{g}{h}$ is determined by a quadratic, whose roots

are always real and of opposite signs. Thus two lines of curvature pass through every point on the surface of the ellipsoid.

Let us now consider the envelope of the surfaces represented by (8).

Differentiating (7) and (8) for f, g, h, we get

$$(b^2-c^2)\frac{u}{f^2}\,df+(c^2-a^2)\frac{v}{g^2}\,dg+(a^2-b^2)\frac{w}{h^2}\,dh=0\ldots\ldots(10),$$

$$df+dg+dh=0\ \ldots\ldots(11).$$

l being an indeterminate factor, we may put

$$l=(b^2-c^2)\frac{u}{f^2},\quad l=(c^2-a^2)\frac{v}{g^2},\quad l=(a^2-b^2)\frac{w}{h^2};$$

whence, taking the values of f, g, h, to substitute them in (8), we deduce

$$\sqrt{b^2-c^2}\,\sqrt{u}+\sqrt{c^2-a^2}\,\sqrt{v}+\sqrt{a^2-b^2}\,\sqrt{w}=0\ldots\ldots(12).$$

As the signs of the radicals are independent, this represents four planes; but c^2-a^2 is negative. Hence the possible part of these planes is their traces on the plane of xz, for which $v=0$. Thus we get the two straight lines

$$\sqrt{b^2-c^2}\,\sqrt{u}+\sqrt{a^2-b^2}\,\sqrt{w}=0\ \ldots\ldots(13),$$

and for the points where they meet the ellipsoid,

$$u+w=1\ \ldots\ldots\ \ldots\ldots\ldots\ldots\ (14),$$

whence

$$u_1=\frac{a^2-b^2}{a^2-c^2},\quad v_1=0,\quad w_1=\frac{b^2-c^2}{a^2-c^2}\ \ldots\ (15).$$

These values belong to the umbilici of the ellipsoid; a result easily anticipated. When they are introduced in (9), it becomes

$$\frac{g^2}{f^2}=0,\quad\text{and similarly}\quad\frac{g^2}{h^2}=0.$$

Hence (8) reduces to

$$v=0\ \ldots\ldots\ldots\ldots\ (16),$$

and represents the principal section of the ellipsoid, which passes through the greatest and least axes. In this case then, as our analysis would lead us to anticipate, the lines of curvature coincide; a result which, although well known, seems not very accurately demonstrated by Leroy. After having shown (p. 309 of the second edition) that the two *directions* of curvature coincide at the umbilical points, he proceeds to integrate, and passes from $\frac{dy}{dx}=0$ to $y=h$, and thence, determining the constant, to $y=0$; which

last represents the line of curvature sought. But $\frac{dy}{dx}$ has been shown to have the value 0, only for the umbilical points, and we are therefore not at liberty to pass by integration from these to any other points at which this may not hold. Were the process legitimate, it would lead to the strange conclusion, that the lines of curvature through an umbilicus are necessarily plane curves.

As there appears to be still some difficulty with regard to the theory of these singular points, we may enquire whether, in order to determine the lines of curvature through any point whatever, more is requisite than to substitute its co-ordinates in the general equation of the lines of curvature, and thus to get two values for the arbitrary constant; whether the result can ever be indeterminate, except when the lines, as at the extremity of an axis of revolution, are so in reality. In this view we see at once, that the process given by Leroy after Poisson for determining the *directions* of curvature at an umbilicus, is simply the ordinary method for ascertaining the position of the branches of any curve at a multiple point; and that the result arrived at, is not that more than two lines of curvature pass through an umbilicus, but that every point which, with reference to the surface, is umbilical, is, with reference to the lines of curvature, a multiple, or more generally a singular point. These suggestions may, perhaps, show how we must determine the lines of curvature which pass through an umbilicus, a problem distinct from that solved by Leroy of finding the directions of curvature.

Many curious properties may be deduced from the equations we have arrived at. Thus, if we take on two concentric and confocal ellipsoids, a series of pairs of corresponding points, (such as are spoken of in the enunciation of Ivory's theorem,) and if the locus of the points on one of the ellipsoids is a line of curvature, then that of those on the other is so too. Again, the traces on the tangent planes at the extremities of the three axes, made by one of the cones represented by (8), are an ellipse and two hyperbolas respectively. The areas of this ellipse, and of the ellipses conjugate to the two hyperbolas, are so related that their continual product is constant for the same ellipsoid, and for all ellipsoids of the same volume. The method of demonstrating these two theorems is so obvious, that it seems unnecessary to enter more fully on either.

It still remains to be shown how we pass from (8) to the projections of the lines of curvature on the co-ordinate planes. The symmetry of the problem is destroyed by the transition; but as it is in this shape that the results are commonly exhibited, we shall dwell rather more upon it than would otherwise have been necessary.

Putting $C = a^2 - b^2$, $B = c^2 - a^2$, $A = b^2 - c^2$, and eliminating u, v, w, successively between (4) and (8), there result

$$\left.\begin{aligned}\left(\frac{C}{h} - \frac{A}{f}\right) u - \left(\frac{B}{g} - \frac{C}{h}\right) v &= \frac{C}{h}\\ \left(\frac{A}{f} - \frac{B}{g}\right) v - \left(\frac{C}{h} - \frac{A}{f}\right) w &= \frac{A}{f}\\ \left(\frac{B}{g} - \frac{C}{h}\right) w - \left(\frac{A}{f} - \frac{B}{g}\right) u &= \frac{B}{g}\end{aligned}\right\} \text{ ...... (17).}$$

Put

$$\frac{C}{h} - \frac{A}{f} = k, \quad \frac{B}{g} - \frac{C}{h} = l, \quad \frac{A}{f} - \frac{B}{g} = m \text{ ...... (18).}$$

Then $kf - lg = C\left(\frac{f+g}{h}\right) - A - B = -(A + B + C) = 0.$

Thus we get the relations

$$\left.\begin{aligned} kf - lg &= 0\\ lh - mf &= 0\\ mg - kh &= 0\end{aligned}\right\} \text{ .... (19).}$$

Hence, $\frac{A}{mf} - \frac{B}{mg} = 1 = \frac{1}{h}\left(\frac{A}{l} - \frac{B}{k}\right)$,

and consequently

$$\left.\begin{aligned} h &= \frac{A}{l} - \frac{B}{k}\\ \text{similarly, } g &= \frac{C}{m} - \frac{A}{l}\\ f &= \frac{B}{k} - \frac{C}{m}\end{aligned}\right\} \text{ .... (20).}$$

By means of (18) and (20) the equations (17) become

$$\left.\begin{aligned} ku - lv &= \frac{Clk}{Ak - Bl}\\ mv - kw &= \frac{Amk}{Bm - Ck}\\ lw - mu &= \frac{Blm}{Cl - Am}\end{aligned}\right\} \text{ ......(21);}$$

which, restoring their values to u, v, w, may be written

$$\frac{y^2}{b^2} = \frac{k}{l}\left\{\frac{x^2}{a^2} - \frac{C}{A\frac{k}{l} - B}\right\}, \text{ \&c. .... (22).}$$

Put $\frac{k}{l}\frac{b^2}{a^2} = m$; $\therefore \frac{k}{l} = m\frac{a^2}{b^2}$, and then we get

$$y^2 = m\left\{x^2 - \frac{a^2}{b^2}\frac{a^2 - b^2}{m\frac{a^2}{b^2}(b^2 - c^2) - (c^2 - a^2)}\right\},$$

$$\text{or} \quad y^2 = m\left\{x^2 - \frac{a^2(a^2 - b^2)}{ma^2(b^2 - c^2) - b^2(c^2 - a^2)}\right\} \ldots (23),$$

with similar equations for the projections on the other co-ordinate planes. This result is identical with the known one in Leroy, p. 304, or Hymers, p. 201.

It is hoped that the novelty of treating symmetrically a non-integrable equation in three variables, will be admitted as an excuse for the length to which this paper has extended itself.

X.—ON THE THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES.

Although it is usual to illustrate the Theory of Maxima and Minima of Functions of one Variable by a reference to the properties of curve lines, I do not remember that any similar illustration has been used in treating of Maxima and Minima of Functions of two Variables. As far as the first condition is concerned, the illustrations will be the same in both cases, but the second or Lagrange's condition in functions of two variables has nothing analogous in the case of two variables. The geometrical explanation gives a very distinct idea of its meaning, and on that account I think that a notice of it may be useful to the student.

Let us briefly consider how the condition arises analytically, and afterwards proceed to the geometrical interpretation of the various steps.

If $z = f(x, y)$ be a maximum or minimum, and z_1 be the value of z when $x + h$, $y + k$ are substituted for x and y, z_2 the value when $x - h$, and $y - k$ are substituted; then for a maximum we must have

$$z_1 < z \text{ and } z_2 < z,$$

$$\text{or} \quad z_1 - z < 0, \quad z_2 - z < 0;$$

and for a minimum,

$$z_1 - z > 0, \quad z_2 - z > 0;$$

Now

$$z_1 - z = \frac{dz}{dx} h + \frac{dz}{dy} k + \frac{1}{1.2}\left(\frac{d^2z}{dx^2} h^2 + 2\frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2\right) + \&c.$$

and

$$z_2 - z = -\left(\frac{dz}{dx} h + \frac{dz}{dy} k\right) + \frac{1}{1.2}\left(\frac{d^2z}{dx^2} h^2 + 2\frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2\right) - \&c.$$

Now either for a maximum or minimum it appears that $z_1 - z$ and $z_2 - z$ must be of the same sign, and as h and k can be assumed so small that the sign of the whole series after them depends on the sign of the terms involving their first powers, and as these are necessarily of opposite signs. In the two series $z_1 - z$ and $z_2 - z$ cannot be of the same sign, unless the terms involving the first powers of h and k vanish, or

$$\frac{dz}{dx} h + \frac{dz}{dy} k = 0,$$

which, as h and k are independent, involves the two conditions

$$\frac{dz}{dx} = 0, \qquad \frac{dz}{dy} = 0 \;\ldots\ldots\ldots\; (A).$$

But $z_1 - z$ and $z_2 - z$ must both of them remain of the same sign, whatever value we assign to h and k, and therefore the first remaining term of the series must remain constantly of the same sign, whatever values we assign to h and k. That is to say, the expression

$$\frac{d^2z}{dx^2} h^2 + 2\frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2,$$

must not pass from $+$ to $-$, or conversely, from changes in the magnitude or sign of h and k. Let $k = mh$, then the expression becomes

$$h^2\left(\frac{d^2z}{dx^2} + 2\frac{d^2z}{dx\,dy} m + \frac{d^2z}{dy^2} m^2\right);$$

and, as h^2 is essentially positive, the sign of

$$\frac{d^2z}{dy^2} m^2 + 2\frac{d^2z}{dx\,dy} m + \frac{d^2z}{dx^2},$$

must not change. Now this expression can only change sign by passing through 0; and, in order that this may never happen from any change in m, the value of m derived from that expression equated to 0 must be impossible.

The solution of the equation gives, if we put

$$\frac{d^2z}{dx^2} = r, \quad \frac{d^2z}{dx\,dy} = s, \quad \frac{d^2z}{dy^2} = t,$$

$$m = -\frac{s}{t} \pm \frac{\sqrt{s^2 - rt}}{t};$$

and, in order that this may be impossible, we must have

$$s^2 < rt,$$

$$\text{or} \quad \left(\frac{d^2z}{dx\,dy}\right)^2 - \left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) = 0 \;\ldots\ldots\; \text{(B)}.$$

This is Lagrange's condition.

Let us now consider the geometrical interpretation of the various steps.

$$z = f(x, y)$$

represents the equation to a surface, and the conditions (A) imply that the tangent plane must be parallel to the plane of xy, since the equation to the tangent plane becomes, under those conditions,

$$z - z_1 = 0.$$

The assumption of $k = mh$ establishing a relation between the increments of x and y, independent of z, corresponds to taking a section of the surface by a plane perpendicular to xy, its trace being inclined to the axis of x at an angle whose tangent is m. For the interpretation of the condition B, we must have recourse to the expression for the radius of curvature of a normal section.

If ρ be the radius of curvature, we have, under the condition that $\frac{dz}{dx} = 0$, $\frac{dz}{dy} = 0$,

$$\rho = \frac{1 + m^2}{r + 2sm + tm^2}.$$

Now the sign of this depends only on that of the denominator, since the numerator equated to zero gives only impossible values for m, but the denominator is exactly the quantity, the sign of which was considered before. Therefore the condition (B), considered geometrically, implies that the sign of the radius of curvature of a normal section shall not change, that is, that every section at the point under consideration must be either convex or concave, but must never pass from one species of curvature to the other. The reason of this is obvious, as the ordinate which is a maximum for a concave section is a minimum for a convex one, and *vice versâ*. It might perhaps be advantageous to have a name appropriated to those points of surfaces for which the conditions (A) hold, but not the condition (B). Such points, though they do not possess the property of being absolute maxima or minima, are yet for many purposes quite as much worthy of attention, since we have frequently to consider geometrically only the fact that the ordinate is stationary for a short space, or that the tangent plane is then perpendicular to the ordinate. Thus, for instance, in investigating the properties of principal diameters in surfaces of the second order, we have only to consider the first condition, and do not require to pay attention to the second. Perhaps the name of "Stationary Points" would be sufficiently distinctive.

D.

XI.—NOTE ON THE CALCULATION OF FORMULÆ IN DIFFRACTION.

THE following transformation is occasionally found useful in calculating the formulæ which occur in the investigation of phenomena of diffraction. These generally depend on finding the value of the definite integral

$$\int_b^a dx\, f(x) \sin \frac{2\pi}{\lambda} \{vt - \psi(x)\},$$

in the form of

$$\mathrm{A} \sin \frac{2\pi}{\lambda} vt + \mathrm{B} \cos \frac{2\pi}{\lambda} vt;$$

when the intensity of illumination is proportional to $\mathrm{A}^2 + \mathrm{B}^2$. Now

$$\mathrm{A} = \int_b^a dx\, f(x) \cos \left(\frac{2\pi}{\lambda} \psi(x)\right), \quad \mathrm{B} = -\int_b^a dx\, f(x) \sin \left(\frac{2\pi}{\lambda} \psi(x)\right);$$

putting for the cosine and sine their exponential values, squaring and adding, we find

$$\mathrm{A}^2 + \mathrm{B}^2 = \left(\int_b^a dx\, f(x)\, \varepsilon^{\frac{2\pi}{\lambda} \psi(x) \sqrt{-1}}\right) \left(\int_b^a dx\, f(x)\, \varepsilon^{-\frac{2\pi}{\lambda} \psi(x) \sqrt{-1}}\right).$$

In practice, $\psi(x)$ is generally and may always be made x; and if one of the limits is ∞, it is to be observed that we must consider $\varepsilon^{x\sqrt{-1}}$ to be equal to 0 when $x = \infty$, that being the average of all its values. In such cases the formula is generally considerably simplified, especially when $f(x)$ is constant.

As an example of the application of this formula, let us take the problem of finding the intensity at the centre of the shadow of a small circular disk.

Let A (fig. 7.) be the centre of the disk, AC its radius $= c$, B the centre of the shadow, AB $= a$, BC $= b$, PB $= r$, AP $= u$.

The vibration at B due to the action of an annulus whose radius is u and distance from B r, is

$$\mathrm{C} \frac{2\pi\, u\, du}{r} \sin \frac{2\pi}{\lambda} (vt - r);$$

and the whole vibration excited at B is given by

$$2\pi \mathrm{C} \int_c^\infty \frac{u\, du}{r} \sin \frac{2\pi}{\lambda} (vt - r).$$

But as $r^2 = a^2 + u^2$, and therefore $u\,du = r\,dr$, this may be changed into

$$2\pi \mathrm{C} \int_b^\infty dr \sin \frac{2\pi}{\lambda} (vt - r),$$

b being the value of r corresponding to $u = c$.

By our previous transformation we find for the intensity of the illumination the expression

$$4\pi^2 C^2 \left(\int_b^\infty dr\, \varepsilon^{\frac{2\pi}{\lambda} r\sqrt{-1}}\right)\left(\int_b^\infty dr\, \varepsilon^{-\frac{2\pi}{\lambda} r\sqrt{-1}}\right),$$

which, on effecting the operations indicated, becomes

$$4\pi^2 C^2 \frac{\lambda}{2\pi\sqrt{-1}}\, \varepsilon^{\frac{2\pi}{\lambda} r\sqrt{-1}} \cdot \frac{-\lambda}{2\pi\sqrt{-1}}\, \varepsilon^{-\frac{2\pi}{\lambda}\sqrt{-1}}$$

$$= 4\pi^2 C^2 \frac{\lambda^2}{4\pi^2} = C^2\lambda^2,$$

which, being independent of the radius of the disk, shews that the intensity is the same as if the disk were removed.

H. T.

XII.—MATHEMATICAL NOTES.

1. The area of a polygon of a given number of sides, circumscribing a given oval figure, will be the least possible when each side is bisected in the point of contact.

This elegant proposition, given in the *Senate-House Problems* for 1836, may be easily demonstrated as follows:—

Let AB, BC, CD, be consecutive sides of the polygon. Produce AB, DC, to meet in E; then BC must, by the condition of the minimum, be in such a position that EBC is a maximum.

Refer the oval to EA, ED, for axes, then the equation to the tangent BC is

$$y'\,dx - x'\,dy = y\,dx - x\,dy,$$

y and x being the co-ordinates of the point of contact P.

Put $x' = 0$;

$$\therefore\ y_0\,dx = y\,dx - x\,dy,$$

$$\text{and so}\quad -x_0\,dy = y\,dx - x\,dy.$$

$$\text{Also area of EBC} = \tfrac{1}{2}x_0 y_0 \sin E.$$

Hence, $\dfrac{(y\,dx - x\,dy)^2}{dx\,dy}$ is a maximum, (the minus sign is immaterial).

Differentiate, considering x as independent; then

$$\frac{y\,dx - x\,dy}{dx\,dy}\, d^2y \left(\frac{y\,dx - x\,dy}{dy} + 2x\right) = 0.$$

The last factor only gives a solution;

$$\therefore\ x = -\tfrac{1}{2}\frac{y\,dx - x\,dy}{dy} = \tfrac{1}{2}x_0,$$

that is, PM being parallel to EC, EM$=\tfrac{1}{2}$EB, and $\therefore$ BP=PC, or BC is bisected in the point of contact P. The same is true of any other side, and therefore every side is bisected in the point of contact.

ε.

2. The following method of investigating the conditions that a straight line and a plane may be at right angles to one another, is easier than the geometrical one which is ordinarily given in books on Analytical Geometry, (see *Leroy's* Géométrie, p. 28).

Let the equation to any plane be

$$Ax + By + Cz = D \ldots\ldots (1),$$

and let the equations to a straight line at right angles to it be

$$\left.\begin{aligned} x &= mz + p \\ y &= nz + q \end{aligned}\right\} \ldots\ldots\ldots (2).$$

Let the equations to any straight line lying within the plane (1), and passing through the intersection of (1) and (2), be

$$\left.\begin{aligned} x &= m'z + p' \\ y &= n'z + q' \end{aligned}\right\} \ldots\ldots\ldots (3).$$

Then, since (2) and (3) must be at right angles to each other, we have

$$1 + mm' + nn' = 0 \ldots\ldots (4).$$

But, since (3) coincides with (1), we have

$$Am' + Bn' + C = 0;$$

and therefore, multiplying the equation (4) by (B), and introducing this last relation, we have

$$B + Bmm' - n(Am' + C) = 0,$$

$$\text{or}\quad B - nC + m'(Bm - An) = 0;$$

and this being true for all values of the indeterminate quantity m', we have, as the required conditions,

$$B - nC = 0, \quad \text{or } n = \frac{B}{C},$$

$$\text{and}\quad Bm - An = 0, \quad \text{or } m = \frac{An}{B} = \frac{A}{C}.$$

w.

3. Let p, p', be two forces into which a given system on a rigid body may be resolved, a, θ, their least distance, and inclination of their directions; $pp'a \sin\theta$ is invariable. (*Senate-House*, 1833.)

Let the line a meet the directions of p and p' in P and P′ respectively. At P apply two forces equal and parallel to p', and

opposite each other. Thus the system of forces is replaced by the couple $p'a$, and by the force at P, which is the resultant of p and p'. Resolve this, the resultant, along the axis of the couple and in its plane. Then the former component can arise only from the resolved part of p, as p' is wholly in the plane of the couple. Also, as the shortest distance is perpendicular to both lines, it follows that the arm of the couple is perpendicular at P to the plane which contains the two forces p and p'. Hence θ, their mutual inclination, is that of p on the plane of the couple, and therefore $p \sin\theta$ is the part of the general resultant resolved along the axis of the couple. Then, if the general resultant makes an angle ϕ with the axis, we have in the usual notation

$$pp'a \sin\theta = \mathrm{GR}\cos\phi = \mathrm{R.G}\cos\phi.$$

Now $\mathrm{G}\cos\phi$, as is known, or as may be easily shown, $= \mathrm{G}_1$, the minimum maximorum moment of the system;

$$\text{therefore } pp'a \sin\theta = \mathrm{G}_1\mathrm{R},$$

which is constant.

ε.

CORRIGENDA.

In Vol. 2, p. 27, line 4, *read* 1 *instead of* (a).

In p. 56, line 14, *should be written*

$$\frac{dy_{\tau+2}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+3}}{dy_{\tau+1}} = \infty, \quad \dots \frac{dy_n}{dy_{\tau+1}} = \infty.$$

In p. 58, line 14 from the bottom, change $\dfrac{dy_{\tau+2}}{dy_\tau}$ into $\dfrac{dy_{\tau+2}}{dy_{\tau+1}}$; and *read, instead of* line 7 from the bottom,

$$\frac{dy_2}{dy_1} = \infty, \quad \frac{dy_3}{dy_2} = \infty, \quad \frac{dy_4}{dy_3} = \infty, \quad \dots \frac{dy_n}{dy_{n-1}} = \infty.$$

In p. 95, line 2, *read* $\mathrm{G}.a^{x-1}\,\Sigma\dfrac{u_x}{a_x}$ *for* $\mathrm{G}.a^{x-1}\,\varepsilon\dfrac{u_x}{a_x}$.

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I.—ON THE EXPRESSIBILITY OF THE ROOTS OF ALGEBRAIC EQUATIONS.

By J. M. PEEBLES.

ASSUMING* the expression

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0,$$

n being a whole positive number, and $a_1, a_2, \dots a_n$, being general symbols quite disconnected with each other; and supposing also that there are only n different expressions which substituted for x render both sides of the above equation identical, and that these n expressions are different values of the same expression, which we shall call (x), and which has no more than these n values, which values also are such, that $(-a_1)$ is equal to their sum, (a_2) is equal to the sum of their products taken two and two, &c., so that the expression $x^n + a_1 x^{n-1} + \dots + a_n$ may be regarded as the product of the n values of the expression $x - (x)$; let us consider farther some of the properties of this hypothetical expression (x).

1. And we observe that (x) must be a function of n detached and independent quantities $\rho_1, \rho_2, \dots \rho_n$, inasmuch as the product of the n values of the expression $x - (x)$ is an expression which contains n absolutely independent and disconnected quantities.

2. This then, like every other property, can only be secured by the form of the function (x), but it may either be stated implicitly or explicitly. It is plain however that the implicit form necessarily implies the existence of an explicit one. Let us call (x') the explicit form, which may either be identical with (x) or different from that function. It must however be such as to *show* the independence and disconnection of n quantities $\rho_1 \dots \rho_n$.

* The Author thinks it necessary to state, that this [illegible] regarded as little more than a sketch of his method of considerin[illegible]

3. Now we observe that if any quantities, as ρ_1 and ρ_2, are subjected to the very same operations, so that the expressions in which they occur are functions of $(\rho_1 + \rho_2)$, then, so far as these operations or expressions are concerned, these quantities are undistinguishable from each other, and must be regarded as forming but one quantity. Hence the n values of (x) must not be such that we may suppose that function to be of the form $\phi(\rho_1 + \rho_2)$ in regard to any two of the quantities $\rho_1, \dots \rho_n$. We must suppose then that there is some method of variation applied to the quantities ρ_1, ρ_2, &c.; and upon this method of variation must depend the establishing of every property of these quantities. Hence it appears that the only way in which the independence of each other of these quantities can be secured, is by the independence of the variations more particularly connected with each of them, of those of the others. And if we suppose that the quantities ρ_1, ρ_2, &c. enter (x') in such a manner that $(x') = \phi(\phi_1\rho_1, \phi_2\rho_2, \dots \phi_n\rho_n)$, the symbols $\phi_1\rho$, $\phi_2\rho$, &c. expressing functions which have various values, (ρ being constant,) and $\phi\rho$ having but one value:, then it is plain, that as the independence and detached nature of the quantities ρ_1, ρ_2, &c., is shewn only by the independence and disconnection of the values of the functions $\phi_1\rho_1$, $\phi_2\rho_2$, &c., the n values of (x'), which are those of (x), must exhaust all the values of (x'), in which the values of $\phi_1\rho_1$, $\phi_2\rho_2$, &c., can occur together in substantially different ways. In fact, (x') must give no new property to the quantities ρ_1, ρ_2, &c., it must only declare explicitly what is implied implicitly in (x). Moreover, it is plain that all the purposes for which we suppose that the functions $\phi_1\rho_1$, $\phi_2\rho_2$, &c., have various values, may be answered by supposing some one of them, as $\phi_1\rho_1$, to be constant. In fact, the independence of the symbols ρ_1 and ρ_2 of each other, may be as well secured by assigning unfettered and independent variations to one only of the functions $\phi_1\rho_1$, $\phi_2\rho_2$, as to both; and then the n values of (x'), which are those of (x), must exhaust the values of (x') in which the relations between the forms in which the quantities $\rho_2, \dots \rho_n$, enter (x) are really different. We shall always suppose this last case, inasmuch as it reduces by one the number of independent sources of variation, and on this account will be seen to be most favourable to the existence of (x).

4. The only source of variation which we now consider arises from the different values of such expressions as $1^{\frac{1}{m}}$, m being of the form $\dfrac{a + a + \text{\&c.}}{a}$. Hence we have only to consider how the independence and disconnection of n quantities in (x), and the other conditions which that function must satisfy, may be secured by radicals.

5. In this investigation we may suppose that there is no term in (x) with a variable denominator, as by multiplying both numerator and denominator by all the values of the denominator we obtain an invariable denominator.

6. It is plain that the detached and independent nature of the

three quantities, ρ_1, ρ_2, ρ_3, is not necessarily secured by the form of the expression $\rho_1\rho_3^{\frac{1}{m}} + \rho_2\rho_3^{\frac{m_1}{m}}$ since by it, wherever ρ_1 occurs multiplied by the $\frac{1}{m}$th root of ρ_3, ρ_2 is also found multiplied by the m_1th power of that root; and this may establish a relation between the quantities ρ_1, ρ_2, ρ_3. In fact, the expressions

$$\rho_1\rho_2^{\frac{1}{m}} = (\rho_1^m\rho_2)^{\frac{1}{m}}, \text{ and } \rho_1^{\frac{1}{m_1}}\rho_2^{\frac{1}{m_2}} = (\rho_1^{m_2}\rho_2^{m_1})^{\frac{1}{m_1 m_2}},$$

are of the forms $\rho_1^{\frac{1}{m}}$ and $\rho_1^{\frac{1}{m_1 m_2}}$ respectively, and are thus in no way distinguishable from them. And the expression

$$\rho_1\rho_3^{\frac{1}{m}} + \rho_2\rho_3^{\frac{m_1}{m}} = (\rho_1^m\rho_3)^{\frac{1}{m}} + (\rho_2^{\frac{m}{m_1}}\rho_3)^{\frac{m_1}{m}},$$

is of the form $\rho_1 1^{\frac{1}{m}} + \rho_2 1^{\frac{m_1}{m}}$, which evidently does not necessarily secure that the symbols ρ_1 and ρ_2 are detached and independent.

7. We go on to show that in order to secure the disconnection of the symbols $\rho_1 \dots \rho_n$, if it can be done at all by means of radicals, $(n - 1)$ independent radicals are needed. The isolated expression $\rho_1 + \rho_2^{\frac{1}{m_1}}$, by means of one radical, secures in the only way in which the employment of radicals can, that the symbols ρ_1 and ρ_2 are independent. And it is plain that the above form comprehends the expressions

$$\rho_3\rho_1 + \rho_3\rho_2^{\frac{1}{m_1}}, \quad \rho_1 + \rho_3 + \dots + (\rho_2 + \rho_4 + \dots)^{\frac{1}{m_1}}.$$

And by the last sec. the form of the expression $\rho_1 + \rho_3\rho_2^{\frac{1}{m_1}} + \rho_4\rho_2^{\frac{m_2}{m_1}}$ does not necessarily secure that three symbols are detached and independent. Hence it appears that one independent radical cannot give a form which secures that more than two quantities are detached and independent.

8. In the case of three independent quantities, we may consider, that as any one of them must be independent of the remaining two, the form by which one of them is related to the remaining two, must be that by which the independence of the two quantities of the last section is secured. Hence the form or forms of this case will be found by substituting for any one of the independent quantities of the last case the form for two independent quantities. We thus find the two forms for (x'),

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}})^{\frac{1}{m_2}}.$$

And by sec. 6, it appears that the two radicals of these expressions cannot produce a form which secures the detached nature of more than three quantities.

9. In the case of four independent quantities, we may consider that two of them must be independent of the other two, while the members of each of these two are ind ach other: and, by the application of the form of sec. lependent quan-

tities, we obtain forms for four independent quantities. And we may consider that any one of the four quantities must be independent of the remaining three, while these three are independent of each other; and, by the application of the forms of sec. 7 and 8, we also obtain forms for four independent quantities. In this manner we obtain the forms for (x'),

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + \rho_4^{\frac{1}{m_3}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}} + \rho_4^{\frac{1}{m_2}})^{\frac{1}{m_3}},$$

$$\rho_1 + \{\rho_2 + (\rho_3 + \rho_4^{\frac{1}{m_1}})^{\frac{1}{m_2}}\}^{\frac{1}{m_3}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + (\rho_3 + \rho_4^{\frac{1}{m_2}})^{\frac{1}{m_3}}.$$

And as it is plain that these are the only forms which we can have, if the detached nature of four quantities can be secured at all by means of radicals it must be by them.

10. Passing on now to the consideration of the case of five independent quantities, it appears, that if by means of radicals this can be secured at all, it can only be by the following forms for (x'):

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + \rho_4^{\frac{1}{m_3}} + \rho_5^{\frac{1}{m_4}}, \quad \rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}} + \rho_4^{\frac{1}{m_2}} + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + \{\rho_2 + (\rho_3 + \rho_4^{\frac{1}{m_1}} + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + [\rho_2 + \{\rho_3 + (\rho_4 + \rho_5^{\frac{1}{m_1}})^{\frac{1}{m_2}}\}^{\frac{1}{m_3}}]^{\frac{1}{m_4}},$$

$$\rho_1 + \{\rho_2 + \rho_3^{\frac{1}{m_1}} + (\rho_4 + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + (\rho_3 + \rho_4^{\frac{1}{m_2}} + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + \rho_2^{\frac{1}{m_1}} + \{\rho_3 + (\rho_4 + \rho_5^{\frac{1}{m_2}})^{\frac{1}{m_3}}\}^{\frac{1}{m_4}}, \quad \rho_1 + \rho_2^{\frac{1}{m_1}} + \rho_3^{\frac{1}{m_2}} + (\rho_4 + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}},$$

$$\rho_1 + (\rho_2 + \rho_3^{\frac{1}{m_1}})^{\frac{1}{m_2}} + (\rho_4 + \rho_5^{\frac{1}{m_3}})^{\frac{1}{m_4}}.$$

And we might proceed in this way to find the forms for the cases of 6, 7, &c. quantities.

11. As irrational quantities can only be rendered rational by involution, and as the sum of the values of (x') which satisfy the equation is a rational quantity $(-a_1)$, it follows that in this sum the radicals must vanish. Hence, referring to the preceding forms of (x'), it is plain that we must have

$$-a_1 = n\rho_1, \text{ and } \therefore \rho_1 = -\frac{a_1}{n}, \text{ and } (x') = -\frac{a_1}{n} + (y'),$$

$$\text{or } (x') + \frac{a_1}{n} = (y').$$

12. If the sum of m_1 values of the expression $1^{\frac{1}{m}}$ vanishes, m is a multiple of m_1; and these m_1 values are those of the expression $1^{\frac{1}{m_1}}$, multiplied by some quantity. (See the note at the end of the paper.) It is easy to see, however, from what follows, that it is not necessary to our argument that we substantiate this theorem in all its generality.

13. Let us now suppose that n is a prime number. Then it is plain that the first of the forms of sec. 7, 8, 9, and 10, alone satisfy the conditions of sec. 11 and 12. For, taking the expres-

sion $(\rho_1 + \rho_2^{\frac{1}{m_1}})^{\frac{1}{m_2}}$, it follows from the last section, that the sum of n values of this expression can vanish only when $m_2 = n$, or $=$ some multiple of n; and then the expression $(\rho_1 + \rho_2^{\frac{1}{m_1}})$ can only have one value, and is therefore of the form ρ_1, and the whole expression is not distinguishable from the expression $\rho_1^{\frac{1}{m_2}}$. Hence, when n is a prime number, we must have

$$(y') = \rho_2^{\frac{1}{n}} + \rho_3^{\frac{1}{n}} + \dots + \rho_n^{\frac{1}{n}}.$$

And by sec. 2, and this form, it appears that if we call (y'_1), $(y'_2) \dots (y'_n)$, the n values of (y') which satisfy the equation, and (y) any one of their number; then, if (y') have more than n values, the values of (y') must be contained in the expression $(y)\, 1^{\frac{1}{n}}$. But considering the above expression for (y'), it is plain that the number of its values is n^{n-1}. Hence, if the above form for (y') has more than n values, we ought to have $n^{n-1} = n^2$, or $n^{n-3} = 1$; which equation can only be satisfied upon the supposition of $n=1$ or $n=3$. Moreover, if the above form has no more than n values, and thus the implicit is also the explicit form, we must have $n^{n-1} = n$ or $n^{n-2} = 1$, which can only be satisfied by making $n = 1$ or $n = 2$.

Hence we conclude that when n is any other prime number than 2 or 3, it is not possible to find an expression whose different values are produced by radicals, which satisfies the conditions which the expression (x) ought to satisfy. Moreover, when n is equal to 2 or 3, we cannot infer the possibility of (x) from the preceding investigation. But it has been long known that when $n = 2$,

$$(x) = (x') = -\frac{a_1}{n} + \rho_1^{\frac{1}{2}},$$

and when $n = 3$,

$$(x') = -\frac{a_1}{3} + \rho_1^{\frac{1}{3}} + \rho_2^{\frac{1}{3}}, \text{ and } (x) = -\frac{a_1}{3} + \rho_1^{\frac{1}{3}} + a\rho_1^{-\frac{1}{3}},$$

which results are quite in harmony with those of the preceding investigation.

Moreover, as (x) is impossible for equations of the fifth degree, it is also impossible for all equations whatever of a higher degree. For, n being greater than 5, if we suppose that (x) is possible for the equation of the n^{th} degree; then, by supposing in (x),

$$a_n = a_{n-1} = a_{n-2} = \dots = a_6 = 0,$$

we should obtain the expression (x) for the equation of the fifth degree, which is impossible.

14. Let us now suppose that n is the product of two prime numbers, n_1 and n_2. Then it is plain that the conditions of sec. 11 and 12 are satisfied by terms of the form

$$\rho_1^{\frac{1}{n_1}}, \quad \rho_1^{\frac{1}{n_2}}, \quad \rho_1^{\frac{1}{n}}, \quad (\rho_1 + \rho_2^{\frac{1}{n_1}})^{\frac{1}{n_2}},$$

only. And by the last section we may confine ourselves to the case of $n_1 = n_2 = 2$, and therefore $n = 4$. The only forms then for (y') in this case, are

$$\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}},\ \rho_2^{\frac{1}{4}} + \rho_3^{\frac{1}{4}} + \rho_4^{\frac{1}{4}},\ \rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{4}},\ \rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{4}} + \rho_4^{\frac{1}{4}},$$
$$\rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}},\ \rho_2^{\frac{1}{4}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}.$$

And, omitting the second of these forms by last section, the condition must be satisfied of $(y') = 1^{\frac{1}{2}}(y)$.

Now the number of values of the above forms are respectively

$$2^3,\quad 2^2 \times 4,\quad 2 \times 4^2,\quad 2^3,\quad 4 \times 2^2;$$

and the number of values of the expression $1^{\frac{1}{2}}(y)$ is 2×4. Hence it is only the forms which correspond to the following expressions which are equal to zero, which can satisfy the conditions of (x') and (x). The expressions are

$$2 \times 4 - 2^3,\quad 2 \times 4 - 2^2 \times 4,\quad 2 \times 4 - 2 \times 4^2,\quad 2 \times 4 - 2^3,$$
$$2 \times 4 - 4 \times 2^2;$$

and as it is only the first and fourth which vanish, the only forms by which (x') can be represented are

$$\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}} \text{ and } \rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}.$$

We cannot, however, infer from this that (x) is possible, as it has not been shown that *all* the conditions of the existence of that function are satisfied. But it is well known, that for the equation of the fourth degree,

$$(x) = -\frac{a_1}{n} + \rho_2^{\frac{1}{2}} + (a + b\rho_2^{\frac{1}{2}})^{\frac{1}{2}},$$

and (x') is of the form

$$-\frac{a_1}{4} + (\rho_2^{\frac{1}{2}} + \rho_3^{\frac{1}{2}} + \rho_4^{\frac{1}{2}}), \text{ or } -\frac{a_1}{4} + \{\rho_2^{\frac{1}{2}} + (\rho_3 + \rho_4^{\frac{1}{2}})^{\frac{1}{2}}\};$$

which is quite in harmony with the preceding results.

Note. The theorem of section 12 may be established as follows.

If we designate by r the expression

$$\cos\frac{2\pi}{m} + \sqrt{-1}\,\sin\frac{2\pi}{m},$$

then $r, r^2, \ldots r^m$, are the m values of $1^{\frac{1}{m}}$. Calling these $\rho_1, \rho_2, \ldots \rho_m$, respectively, let us suppose that

$$\rho_1 + \rho_p + \rho_q + \cdots + \rho_u + \rho_v = 0 \ldots\ldots (1),$$

p being > 1, $q > p$, $v > u$, $m + 1 > v$.

The distances between the successive terms of the above expression, or the numbers of terms of the series $\rho_1, \rho_2, \rho_3, \ldots \rho_m$, intervening betwixt any successive two of them, may either be equal or unequal. If equal, we must have

$$p - 1 = q - p = \&c. - q = \&c. = u - \&c. = v - u = m + 1 - v \ldots (2).$$

Let us suppose that the preceding equations are not satisfied, and consider the consequences of this hypothesis.

From (1), conjoined with the consideration of the connection between ρ_1 and r, we deduce the set of equations

$$\left.\begin{array}{l} \rho_1 + \rho_p \quad + \rho_q \quad + \cdots + \rho_u \quad + \rho_v \quad = 0, \\ \rho_2 + \rho_{p+1} + \rho_{q+1} + \cdots + \rho_{u+1} + \rho_{v+1} = 0, \\ \rho_3 + \rho_{p+2} + \rho_{q+2} + \cdots + \rho_{u+2} + \rho_{v+2} = 0, \\ \cdots \quad \cdots \quad \cdots \\ \rho_{m+1-v} + \rho_{m+p-v} + \rho_{m+q-v} + \cdots + \rho_{m+u-v} + \rho_m = 0, \end{array}\right\} \cdots (3);$$

which equations, it is evident, are all different from each other. In the same manner we find the equation

$$\rho_1 + \cdots + \rho_{m+2-v} + \rho_{m+p+1-v} + \rho_{m+q+1-v} + \cdots + \rho_{m+u+1-v} = 0;$$

and from it, as from (1), we derive a second set of equations,

$$\left.\begin{array}{l} \rho_1 + \cdots + \rho_{m+2-v} + \rho_{m+p+1-v} + \rho_{m+q+1-v} + \cdots + \rho_{m+u+1-v} = 0, \\ \rho_2 + \cdots + \rho_{m+3-v} + \rho_{m+p+2-v} + \rho_{m+q+2-v} + \cdots + \rho_{m+u+1-v} = 0, \\ \cdots \quad \cdots \\ \rho_{-u+v} \quad + \rho_{m-u+1} + \rho_{m-u+p} \quad + \rho_{m-u+q} \quad + \cdots + \rho_m \qquad = 0, \end{array}\right\} \cdots (4),$$

all of which are different from each other and from those of (3). Proceeding in this way, we find a third and a fourth &c. set, the equations of which are all different from each other. And thus we proceed until we come to a set precisely the same as the first set. It is plain that the total number of these equations is equal to m. Hence we have m linear equations all different from each other, between the m quantities $\rho_1, \ldots \rho_m$.

Let us consider the consequences of this. And we observe, that if we would eliminate all the quantities except one, contained in any equation, from the nature of the equations it must plainly be by the addition and subtraction of some of the other equations, so that the quantity which we wish to obtain can rise no higher than the first power. Hence it can only have one value. But the values $\rho_1 = 0$, $\rho_2 = 0 \ldots \rho_m = 0$, satisfy the said equations. Hence these are the only values of those quantities which do so.

Considering now the equations

$$\rho_1 = 0, \quad \rho_{m-1} = 0, \quad \text{or} \quad \cos\frac{2\pi}{m} + \sqrt{-1}\sin\frac{2\pi}{m} = 0,$$

$$\cos\frac{2\pi}{m} - \sqrt{-1}\sin\frac{2\pi}{m} = 0,$$

by adding and subtracting we obtain the two simultaneous equations

$$\cos\frac{2\pi}{m} = 0, \qquad \sin\frac{2\pi}{m} = 0;$$

which is impossible. Hence the existence of (1) requires that (2) be satisfied.

Let us then adopt this hypothesis, and call m_1 the number of terms of (1). We then find by (2),

$$q = 2p - 1, \ldots u = (m_1 - 2)p - (m_1 - 3), \quad v = (m_1 - 1)p - (m - 2),$$
$$m_1 p - (m_1 - 1) = m + 1;$$

from the last of which equations we have

$$m_1(p-1)=m.$$

Hence m_1 must be a submultiple of m; and if m be a prime number, $p=2$ and $m_1=m$. Moreover, equation (1) may be written in virtue of the preceding equation,

$$\cos\frac{2\pi}{m_1(p-1)}+\sqrt{-1}\sin\frac{2\pi}{m_1(p-1)}+\cos\frac{2p\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2p\pi}{m_1(p-1)}+\cos\frac{2(2p-1)\pi}{m_1(p-1)}+\sqrt{-1}\sin\frac{2(2p-1)\pi}{m_1(p-1)}+$$
$$\ldots+\cos\frac{2\{(m_1-1)p-(m_1-2)\}\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2\{(m_1-1)p-(m_1-2)\}\pi}{m_1(p-1)}=0,$$

$$\text{or}\cos\frac{2\pi}{m_1(p-1)}+\sqrt{-1}\sin\frac{2\pi}{m_1(p-1)}+\cos\frac{2(p-1)\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2(p-1)\pi+2\pi}{m_1(p-1)}+\cos\frac{2(2p-2)\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2(2p-2)\pi+2\pi}{m_1(p-1)}+\cos\frac{2\{(m_1-1)p-(m-1)\}\pi+2\pi}{m_1(p-1)}$$
$$+\sqrt{-1}\sin\frac{2\{(m-1)p-(m-1)\}\pi+2\pi}{m_1(p-1)}=0,$$

$$\text{or}\left(\cos\frac{2\pi}{m_1}+\sqrt{-1}\sin\frac{2\pi}{m_1}+\cos\frac{4\pi}{m_1}+\sqrt{-1}\sin\frac{4\pi}{m_1}+\right.$$
$$\left.\ldots+\cos\frac{2m_1\pi}{m_1}+\sqrt{-1}\sin\frac{2m_1\pi}{m_1}\right)$$
$$\times\left(\cos\frac{2\pi}{m_1(p-1)}+\sqrt{-1}\sin\frac{2\pi}{m_1(p-1)}\right)=0,$$

$$\text{or}\cos\frac{2\pi}{m_1}+\sqrt{-1}\sin\frac{2\pi}{m_1}+\cos\frac{4\pi}{m_1}+\sqrt{-1}\sin\frac{4\pi}{m_1}+$$
$$\ldots+\cos\frac{2m_1\pi}{m_1}+\sqrt{-1}\sin\frac{2m_1\pi}{m_1}=0.$$

Hence the m_1 values of $1^{\frac{1}{m}}$ constituting the terms of (1), are the m_1 values of $1^{\frac{1}{m_1}}$ multiplied by some quantity which we perceive to be $\left(\cos\frac{2\pi}{m}+\sqrt{-1}\sin\frac{2\pi}{m}\right)$.

Dunfermline, Jan. 1839.

II.—ON THE TAUTOCHRONE IN A RESISTING MEDIUM.

OUR object is to reduce the problem of the tautochronous curve, when the resistance is equal to kv^2 or to $hv + kv^2$, to the cases in which it is a cycloid, viz. when the resistance is equal to zero or to hv.

In vacuo

$$\sqrt{2g}\, t_1 = \int_0^{s_1} \frac{ds}{\sqrt{x_1 - x}},$$

and the necessary and sufficient condition of tautochronism is $s^2 = Ax$. Hence generally

$$\int_0^{z_1} \frac{dz}{\sqrt{Fz_1 - Fz}}$$

is independent of z_1, provided $z^2 = A.Fz$. Now, when $R = kv^2$, there is

$$\frac{d.vdv}{ds} - kv^2 = -g\frac{dx}{ds},$$

$$\text{therefore } e^{-2ks} v^2 = -2g\int e^{-2ks} \frac{dx}{ds}\, ds.$$

$$\text{Let } fs = \int_0 e^{-2ks} \frac{dx}{ds}\, ds,$$

$$\text{therefore } e^{-2ks} v^2 = 2g\,(fs_1 - fs),$$

$$\text{and } \sqrt{2g}\, t_1 = \int_0^{s_1} \frac{e^{-ks}\, ds}{\sqrt{fs_1 - fs}}.$$

Put $dz = e^{-ks} ds$, and let z and s be equal to zero together,

$$\text{therefore } z = \frac{1}{k}(1 - e^{-ks}), \quad fs = Fz,$$

$$\text{and } \sqrt{2g}\, t_1 = \int_0^{z_1} \frac{dz}{\sqrt{Fz_1 - Fz}}.$$

Therefore for tautochronism

$$z^2 = A.Fz.$$

$$\text{But } Fz = \int_0 \frac{dx}{ds}(1 - kz)\, dz,$$

$$\text{therefore } \frac{2}{A} z = az = \frac{dx}{ds}(1 - kz),$$

$$\text{therefore } k\frac{dx}{ds} = a\frac{1 - e^{-ks}}{e^{-ks}},$$

$$\text{or } k\frac{dx}{ds} = a(e^{ks} - 1) \;\ldots\ldots\ldots\; (1),$$

the equation of the tautochronous curve.

We shall have, if $t = 0$ when $s = s_1$,

$$z = z_1 \cos\sqrt{ag}\, t.$$

$$\text{Hence } \frac{d^2s}{dt^2} - k\frac{ds^2}{dt^2} = -ag\frac{e^{ks} - 1}{k}$$

must become, when s is expressed in z,

$$\frac{d^2z}{dt^2} = -agz,$$

a result easily verified, for

$$ds = \frac{1}{1 - kz}dz \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (\alpha).$$

$$d^2s = \frac{1}{1 - kz}d^2z + \frac{k}{(1 - kz)^2}dz^2 \;\ldots\ldots\; (\beta).$$

The coefficient of dz in (α) is of course that of d^2z in (β), which shews that if the equation of motion were

$$\frac{d^2s}{dt^2} + h\frac{ds}{dt} - k\frac{ds^2}{dt^2} = -ag\frac{e^{ks} - 1}{k},$$

$$\text{or } R = hv + kv^2,$$

the equation in z would be

$$\frac{d^2z}{dt^2} + h\frac{dz}{dt} = -agz.$$

This is precisely the form of the equation of motion on a cycloid when $R = hv$. And as in that case t_1 deduced from it is independent of s_1, so in this it will be independent of z_1, and consequently of s_1. Hence the curve whose equation is (1), is tautochronous, not only when $R = kv^2$, but also when $R = hv + kv^2$.

For Laplace's abstruse solution, see the first book of the *Mécanique Celeste*, or Mr. Whewell's *Dynamics*.

R. L. E.

III.—ON THE GENERAL THEORY OF MULTIPLE POINTS.

By W. WALTON, B.A., Trinity College.

1. IT is my object in the present paper to inquire into the nature of those properties of the curvilinear loci of any rational algebraic equation $f(x, y) = 0$, between two variables x and y, which correspond for particular values of x and y to one or both of the peculiar relations $\frac{d}{dx}f(x, y) = 0$, $\frac{d}{dy}f(x, y) = 0$. The geometrical peculiarities, defined by the appellation of 'multiple points,' 'cusps,' and 'isolated' or 'conjugate points,' which writers on the theory of curves have appropriated to the explanation of these singular relations when existing simultaneously, have been hitherto universally regarded as three essentially distinct and unconnected properties. Indeed, 'isolated' or 'conjugate points' have always been regarded as their two names, the one with relation to the curve and the other with reference to the algebraic equation, evidently imply, as points, although actually represented by the equation, yet utterly detached from its curvilinear locus. The supposition of any essential distinction between the natures of these three points, as will be evident from the light in which the subject will be presented in the following pages, has arisen frôm an inadequate appreciation of the general theory of the interpretation of algebraic equations between two variables. I shall endeavour, by the aid of a wider method of viewing the subject, to shew that both 'conjugate points' and 'cusps' are virtually 'multiple points' under different states of manifestation. I shall likewise, before proceeding to the discussion of the coexistence of the two singular relations $\frac{d}{dx}f(x, y) = 0$, $\frac{d}{dy}f(x, y) = 0$, demonstrate that the satisfaction of either of them singly corresponds in all cases to a multiple point of determinable multiplicity.

2. Let the equation

$$f(x, y) = 0 \ldots\ldots\ldots (\mathrm{I})$$

designate the perpetual relation subsisting between the simultaneous co-ordinates of a system of points, the function $f(x, y)$ being supposed to be devoid of radicals and of negative indices, and of μ and ν dimensions respectively in x and y. Then, if we give to x an indefinite number of consecutive values comprehended under the general form $+^{\phi(\alpha)}\alpha$, where $\phi(\alpha)$ denotes any assigned invariable function of α, by ascribing to α every variety of positive and negative magnitude we shall obtain correspondingly from the equation (I) ν indefinite series of values for y of the general form $+^{s}\beta$, where s and β for each series are some particular functions of α. Thus we see, that under these prescribed laws of variation in the magnitude and the affection of x, the equation (I) will determine

ν interminable curves. Again, if we were to make y the regulative symbol in the equation (1) instead of x, it is clear in the same way that we should define the loci of μ interminable curves.

The method of constructing the infinite number of the pairs of conjugate axes which I have adopted in the article on the General Theory of the Interpretation of Equations between two Variables in Algebraic Geometry in the last number of this Journal, I shall continue to employ in the following disquisitions. It is clear then that supposing x to be the regulative symbol in the equation (1), we shall have generally for any assigned value of x' in the equivalent quantitive equations, ν different pairs of values of y' and z', where x', y', z', represent the quantitive co-ordinates of any point of which the affectional ones are x, y. In the same way, if y be chosen as the regulative symbol, we shall have generally for any assigned value of y', μ different pairs of values of x', z'.

In the article of the last number of the Journal to which I have referred, $+^r a$ is taken as the general form of the values of the regulative symbol, r being considered a constant quantity; it is better however to adopt the form $+^{\phi(a)} a$ for this purpose, as being evidently both more general and equally well adapted to the demonstration of the two propositions which I have there discussed.

3. In order to fix upon the mind distinctly the principles which we have laid down above, we will occupy ourselves in the determination of the quantitive equations corresponding to the affectional equation

$$y^2 = x^3 \text{ .... } (1),$$

both when x and when y is chosen as the regulative symbol.

First, let us take x as the regulative symbol, and put it equal to $+^{\phi(a)}. a$, the corresponding values of y being represented by the general expression $+^s\beta$, and we have

$$+^{2s}\beta^2 = +^{3\phi(a)}a^3,$$

and therefore

$$(\cos 4s\pi + -^{\frac{1}{2}} \sin 4s\pi)\,\beta^2 = \{\cos . 6\phi(a)\,\pi + -^{\frac{1}{2}} \sin . 6\phi(a)\,\pi\}\, a^3;$$

and this equation manifestly resolves itself into the two following,

$$\beta^2 \cos 4s\pi = a^3 \cos . 6\phi(a)\,\pi \text{ ...... } (2),$$

$$\beta^2 \sin 4s\pi = a^3 \sin . 6\phi(a)\,\pi \text{ .. ... } (3).$$

But if x', y', z', denote the three quantitive co-ordinates of any point $x = +^{\phi(a)}. a$, $y = +^s\beta$, we have (see Vol. II. No. IX. Art. 3. of this Journal,)

$$x' = a \cos . 2\phi(a)\,\pi \text{ ... .} (4),$$

$$y' = \beta \cos 2s\pi \text{ ....} (5),$$

$$z' = a \sin . 2\phi(a)\,\pi + \beta \sin 2s\pi \text{ ......} (6);$$

and from the five equations (2), (3), (4), (5), (6), we should have to eliminate a, s, β, in order to arrive at the two quantitive equations corresponding to any one of the curves represented by the affectional equation (1).

For the sake of simplicity, suppose $\phi(a) = 0$, and we shall then have, in place of the five general equations, the five following ones,

$$\beta^2 \cos 4s\pi = a^3, \quad \beta^2 \sin 4s\pi = 0,$$
$$x' = a, \quad y' = \beta \cos 2s\pi, \quad z' = \beta \sin 2s\pi.$$

From the second of these we see that $4s\pi = \lambda\pi$, where λ denotes any integral number positive or negative, and therefore $2s\pi = \dfrac{\lambda\pi}{2}$. First let $\lambda = 2m$, any even number, and we have

$$\beta^2 = a^3, \quad x' = a, \quad (-)^m y' = \beta, \quad z' = 0,$$

and therefore, considering that $(-)^{2m} y'^2 = y'^2$, we get as the equations to the corresponding curve

$$y'^2 = x'^3, \quad z' = 0 \;\ldots\ldots\ldots\ldots\ldots\; (7).$$

Secondly, let $\lambda = 2m + 1$, any odd number, and we have

$$-\beta^2 = a^3, \quad x' = a, \quad y' = 0, \quad (-)^m z' = \beta,$$

and therefore, since $(-)^{2m} z'^2 = z'^2$, we have

$$-z'^2 = x'^3, \quad y' = 0 \ldots\; \ldots\ldots (8).$$

Thus we see, that for the peculiar value which we have assigned to the function $\phi(a)$, the affectional equation (1) is equivalent to the two quantitive equations (7) and (8); and it is evident that if we give to x' any value whatever, we shall get from (7) and (8) two pairs of values of y', z', generally different, or that a plane parallel to that of y', z', will be pierced in two points by the curvilinear loci of the equation (1).

Next, take y as the regulative symbol, and for the sake of avoiding operose eliminations, assume it to be equal to β, a symbol of mere magnitude affected by $+$ or $-$. Then the equations between which the elimination of r, a, β, are to be effected, $+^r a$ being the general value of x, will be

$$\beta^2 = a^3 \cos 6r\pi, \quad 0 = a^3 \sin 6r\pi,$$
$$x' = a \cos 2r\pi, \quad y' = \beta, \quad z' = a \sin 2r\pi.$$

From the second of these equations we see that $6r\pi = \lambda\pi$, and therefore $2r\pi = \dfrac{\lambda\pi}{3}$, where λ denotes any integral number, positive or negative. And if we assign to λ each of the three forms $3m$, $3m+1$, $3m+2$, we shall clearly exhaust all its values. First then, let $\lambda = 3m$, and we have

$$\beta^2 = (-)^m a^3, \quad (-)^m x' = a, \quad y' = \beta, \quad z' = 0,$$

and therefore, since $(-)^{4m} x'^3 = x'^3$, we have

$$y'^2 = x'^3, \quad z' = 0 \;\ldots\ldots\ldots\ldots\ldots\; (9).$$

Secondly, let $\lambda = 3m + 1$, and we have

$$\beta^2 = (-)^{m+1} a^3, \quad (-)^m x' = \frac{a}{2}, \quad y' = \beta, \quad (-)^m z' = a\frac{\sqrt{3}}{2},$$

and therefore, since $(-)^{4m+1}x'^3 = -x'^3$, we have

$$y'^2 = -8x'^3, \quad z' = x'\sqrt{3} \ldots\ldots\ldots\ldots (10).$$

Thirdly, let $\lambda = 3m + 2$, and we have

$$\beta^2 = (-)^m a^3, \quad (-)^{m+1}x' = \tfrac{1}{2}a, \quad y' = \beta, \quad (-)^m z' = a\frac{\sqrt{3}}{2},$$

and therefore

$$y'^2 = -8x'^3, \quad z' = -x'\sqrt{3} \ldots\ldots (11).$$

From the three pairs of quantitive equations (9), (10), (11), it is evident that generally a plane drawn parallel to that of x', z', will be pierced in three different points by the curvilinear loci of the equation (1) when y is chosen as the regulative symbol. Like conclusions would have resulted from the imposition of any other value upon the function $\phi(a)$. Thus the equation (1) represents two or three interminable curves accordingly as x or y is the regulative symbol.

4. Suppose that, from the peculiar constitution of the function $f(x, y)$, when we put $x = +^{\phi(a')}.a'$, where a' is some particular value of a, the equation $f(x, y) = 0$ has n equal values $+^{s'}\beta'$ of y, s' and β' being the corresponding particular values of s and β. Consider x as the regulative symbol. In this case it is clear that the curves which belong to x as the regulative symbol, which for the convenience of a technicality we shall call the 'x curves', have a multiplicity at the point $x = +^{\phi(a')}.a'$, $y = +^{s'}\beta'$, of the n^{th} degree. For the sake of brevity put $+^{\phi(a')}.a' = a$ and $+^{s'}\beta' = b$.

By Taylor's theorem, which by the nature of the function $f(x, y)$ is evidently applicable to the present case, we have

$$\begin{aligned} 0 = f(x, y) &= f(a + x - a,\ b + y - b) \\ &= f(a, b) + \left\{(x-a)\frac{d}{da} + (y-b)\frac{d}{db}\right\} f(a, b) \\ &\quad + \frac{1}{1.2}\left\{(x-a)\frac{d}{da} + (y-b)\frac{d}{db}\right\}^2 f(a, b) \\ &\quad + \ldots\ldots\ldots\ldots \end{aligned}$$

But since when we put $x = a$, the equation $f(x, y) = 0$ is to have n equal values b of y, it is clear that $(y - b)^n$ must be a divisor of $f(a, y)$, and therefore, as is evident from the development of $f(x, y)$ which we have just given, of

$$f(a, b) + (y-b)\frac{d}{db}f(a, b) + \frac{1}{1.2}(y-b)^2\frac{d^2}{db^2}f(a, b) + \ldots\ldots$$

Hence obviously the necessary and sufficient conditions for the existence of multiplicity of the n^{th} degree in the 'x curves' at the point $x = a$, $y = b$, are the following relations,

$$f(a, b) = 0, \quad \frac{d}{db}f(a, b) = 0, \quad \frac{d^2}{db^2}f(a, b) = 0,$$

$$\ldots\ldots \frac{d^{n-1}}{db^{n-1}}f(a, b) = 0 \ldots\ldots\ldots (\text{II}).$$

Again, in precisely the same way we might shew, that if the 'y curves' have a multiplicity of the m^{th} degree at the same point $x = a$, $y = b$, for any assignable form of the affectional coefficient of the general expression for y, we must have in addition to the above relations the $m - 1$ following ones,

$$\frac{d}{da}f(a, b)=0, \quad \frac{d^2}{da^2}f(a, b)=0, \ldots \frac{d^{m-1}}{da^{m-1}}f(a, b)=0 \ldots\ldots \text{(III.)}$$

The coexistence of multiplicity, both for the 'x' and the 'y curves' at one and the same point, we shall characterize by the name of 'complex multiplicity', in distinction from the single multiplicity of either the 'x' or the 'y curves', which we shall call 'simple multiplicity'. If there be simple multiplicity of the m^{th} order in the 'x curves', or of the n^{th} order in the 'y curves', at any point, we shall call it multiplicity of the $m_x{}^{\text{th}}$ or $n_y{}^{\text{th}}$ order respectively; and if there be complex multiplicity consisting of two such simple multiplicities, we shall call it multiplicity of the $(m_x + n_y)^{\text{th}}$ order.

5. From what we have said in the preceding section, it is obvious that the forms of which the equation (I) is necessarily and sufficiently susceptible for the existence of multiplicity of the $m_x{}^{\text{th}}$, $n_y{}^{\text{th}}$, $(m_x + n_y)^{\text{th}}$, orders respectively at a point $x=a$, $y=b$, are

$$(x - a)^m \phi(x, y) + (y - b)\chi(x, y) = 0,$$
$$(x - a)\phi(x, y) + (y - b)^n\chi(x, y) = 0,$$
$$(x-a)^m \phi(x, y) + (x-a)^p(y-b)^q \chi(x, y) + (y-b)^n\psi(x, y) = 0,$$

where the quantities $\phi(a, b)$, $\chi(a, b)$, $\psi(a, b)$, are not any of them in any of the three equations equal to zero. Thus it will frequently happen, that the existence of multiple points as well as the order of their multiplicity is manifest on inspection. Take for instances the following equations,

$$ay^2 - x^3 - bx^2 = 0 \ldots\ldots\ldots (1),$$
$$y^5 + ax^4 - b^2xy^2 = 0 \ldots\ldots\ldots (2),$$
$$(x^2 + y^2)^3 - 4a^2x^2y^2 = 0 \ldots\ldots\ldots (3),$$
$$(a^2 + x^2)^2 + y^2 = 0 \ldots\ldots\ldots (4).$$

In (1), when $x = 0$, y^2 is a factor, and when $y = 0$, x^2 is a factor. Hence there is necessarily a complex multiple point at the origin of the $(2_x + 2_y)^{\text{th}}$ order. In (2), when $x = 0$, y^5 is a factor, and when $y = 0$, x^4 is a factor; hence there is at the origin a complex multiple point of the $(4_x + 5_y)^{\text{th}}$ order. In (3), when $x = 0$, y^6 is a factor, and when $y = 0$, x^6 is a factor. Hence there is at the origin a multiple point of the $(6_x + 6_y)^{\text{th}}$ order. In (4), when $x = \pm -^{\frac{1}{2}}a$, y^2 is a factor, and when $y = 0$, $(x + -^{\frac{1}{2}}a)^2$ or $(x - -^{\frac{1}{2}}a)^2$ is a factor. Hence there are two multiple points of the order $2_x + 2_y$, whose co-ordinates are $--^{\frac{1}{2}}a$, 0, and $-^{\frac{1}{2}}a$, 0.

6. One of the most interesting cases of simple multiplicity is

that which presents itself under the form of a maximum or minimum value of y, in a portion of the curvilinear locus of the equation (I), which lies in the plane $+$, $+$, at a point whose tangent is parallel to the axis of x. In this case it is clear that y being taken as the regulative symbol, and receiving every degree of positive and negative magnitude, two values of x will become equal at the point in question. Let $x = a$, $y = b$, be the point: then, from what has been said in section (4), we have

$$\frac{d}{da}f(a, b) = 0 \text{ .... } (1).$$

But we have by differentiating (I) at the point

$$\frac{d}{da}f(a, b) + \frac{d}{db}f(a, b)\frac{db}{da} = 0 \text{ .... } (2),$$

and, the multiplicity being confined to the 'y curves,' $\frac{d}{db}f(a, b)$ has necessarily some finite value. Hence from (2)

$$\frac{db}{da} = 0.$$

If in addition to the multiplicity at the point a, b, which belongs to the portion of the curvilinear locus of the 'y curves' in the plane $+$, $+$, there exist likewise multiplicity at this point in respect to the 'y curves' which may pass through it without lying within the plane, then we shall have, in addition to the relation (1), the $n - 2$ following relations,

$$\frac{d^2}{da^2}f(a, b) = 0, \quad \frac{d^3}{da^3}f(a, b) = 0, \ldots \frac{d^{n-1}}{da^{n-1}}f(a, b) = 0.$$

It is necessary here to remark, that the n is necessarily an even number. For by the theory of equations we know that roots of the form $+^r a$, where r is a fractional quantity, enter equations whose coefficients are merely positive or negative magnitudes by pairs; and therefore when we assign to y various values which are confined to positive and negative magnitude in the neighbourhood of the point a, b, it is clear that we must have for x an even number of values of the form $+^r a$. Hence at the point a, b, an even number of values of x, generally of the form $+^r a$, must degenerate into the form $+^0 a$ or a. And there are, by the nature of the case, only two of the form a for the neighbouring values of y.

Hence, by perpetually differentiating (2) with respect to a for $n - 1$ times, it is successively obvious that

$$\frac{d^2b}{da^2} = 0, \quad \frac{d^3b}{da^3} = 0, \text{ ...... } \frac{d^{n-1}b}{da^{n-1}} = 0;$$

and that finally

$$\frac{d^n}{da^n}f(a, b) + \frac{d}{db}f(a, b)\frac{d^nb}{da^n} = 0;$$

$$\text{whence } \frac{d^n b}{da^n} = - \frac{d^n}{da^n} f(a, b) : \frac{d}{db} f(a, b).$$

Thus we see, that for the existence of a maximum or a minimum value of y, and a tangent parallel to the axis of x, at any point in a portion of the curvilinear locus of the equation (1) which lies within the plane $+$, $+$, an odd number only of its differential coefficients vanish, the order of the first which does not vanish corresponding to the order of the simple multiplicity of the 'y curves' which pass through the point.

It may be remarked, that both Fermat in his *Treatise de Max. et Min.*, and Hudde (see Schooten's *Additions to Des Cartes' Geometry*, at the end of the first volume of his edition), who were the earliest writers on the analytical theory of maxima and minima, founded their reasonings on the idea of two equal ordinates of a curve uniting into one; which agrees with the method of considering the subject which we have developed above. The principle from which Newton deduced the rule for the determination of a maximum or minimum point is, that the fluxion or increment of the ordinate vanishes at the point; and although this method, with certain modifications of expression, has been generally adopted, and is of course perfectly legitimate in itself, yet since it is not comprehensive of the theory of a plurality of branches, which we have shewn to be connected with the occasional vanishing of the differential coefficients of the higher orders at a maximum or minimum point, the solution of the problem according to the notion of Fermat and Hudde seems to deserve a preference.

Ex. As an instance of simple multiplicity, we may take the equation
$$y = (x - a)^4.$$

Four 'y curves' pass through the point $x = a$, $y = 0$; two of which pierce the plane $+$, $+$, without lying within it on either side of the point, while the other two leave the plane on one side only of the point. Only one 'x curve' passes through the point.

7. We will now proceed to explain the true nature of a 'conjugate' or 'isolated' point. Suppose that $x = a$, $y = b$, are the equations to a conjugate point, a and b being both of the form $+^0\rho$. Then, when x receives any value whatever of the form $+^0\rho$ differing slightly from a, y by the nature of a conjugate point will assume n values of the form $+^s\rho$, if n be the degree of the equation in y, where s is some fractional quantity; and from this it is evident, conversely, that if y receives any value whatever of the form $+^0\rho$, differing slightly from b, x will necessarily assume m values of the form $+^s\rho$, if m be the degree of the equation in x, where s is some fractional quantity. Now since roots of the form $+^s\rho$ present themselves by pairs in equations whose coefficients are of the form $+^0\rho$, it is clear that if any one root of the form $+^s\rho$ degenerates into one of the form $+^0\rho$, from any alteration in the magnitudes of the coefficients, their form being unaltered, an

even number of roots must do so: and thus we see clearly, that when x is put $= a$ in the equation (I) at a conjugate point, the resulting equation in y must have a common factor of the form $(y - b)^{2p}$, where p denotes some positive integer; and conversely when we put $y = b$, the resulting equation in x must have a common factor of the form $(x - a)^{2q}$, where q denotes some positive integer. Thus we see that a conjugate point is merely a complex multiple point corresponding to the 'x' and the 'y curves', which belong respectively to the regulative forms $+^0\alpha$ and $+^0\beta$ of x and y, the regulative symbols; the branches of the curves piercing the plane $+$, $+$, at the point. The multiplicity of the point is to be determined according to the principles of section (4). We have been speaking above as if the conjugate point were perfectly out of the way of any branch in the plane $+$, $+$; but it of course may happen that one or more branches in this plane may pass through the point, and under these circumstances it is plain that the plurality of the equal roots a and b of $f(x, b) = 0$ and $f(a, y) = 0$ respectively, will be augmented according to the number of these additional branches.

Ex. There is a conjugate point $x = 0$, $y = 0$, connected with the equation

$$x^2 + y^2 = 0.$$

The equations to the two 'x curves' are

$$x' + z' = 0, \quad y' = 0,$$

$$\text{and } x' - z' = 0, \quad y' = 0.$$

And the equations to the two 'y curves' are

$$y' + z' = 0, \quad x' = 0,$$

$$\text{and } y' - z' = 0, \quad x' = 0,$$

both the x and the y, as regulative symbols, being restricted to the form $+^0\rho$.

Ex. Take the equation

$$y^4 - (x - a)^3 = 0.$$

Here $(a, 0)$ are the co-ordinates of a conjugate point of the $(2_x + 2_y)^{\text{th}}$ order, and of an additional multiplicity of the $(1_x + 2_y)^{\text{th}}$ order; this latter multiplicity corresponding evidently to a minimum value of y.

8. It occasionally happens that the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$, to whatever order it may be, have values of the form $+^0\rho$ at a conjugate point. For instance, the curve whose equation is

$$(y - cx^3)^2 = (x - b)^5 \,.\, (x - a)^6,$$

$$\text{or } y = cx^3 \pm (x - b)^{\frac{5}{2}} \cdot (x - a)^3,$$

where a is less than b, has a conjugate point whose co-ordinates are $x = a$, $y = ca^3$; and differentiating, we have

$$\frac{dy}{dx} = 3cx^2 \pm 3(x-b)^{\frac{5}{2}}(x-a)^2 \pm \frac{5}{2}(x-b)^{\frac{3}{2}}(x-a)^3,$$

$$= 3ca^2 \text{ at the conjugate point,}$$

$$\frac{d^2y}{dx^2} = 6cx \pm 6(x-b)^{\frac{5}{2}}(x-a) \pm 15(x-b)^{\frac{3}{2}}(x-a)^2 \pm \frac{15}{4}(x-b)^{\frac{1}{2}}(x-a)^3,$$

$$= 6ca \text{ at the point.}$$

We will proceed to investigate generally the geometrical peculiarities of such points corresponding to these peculiar algebraical relations.

Take x as the regulative symbol, and assume $+^0\alpha$, or, which is the same thing, α as its general form, $+^s\beta$ being the corresponding form for y.

Then we have

$$\frac{dy}{dx} = \frac{d}{dx}(+^s\beta)$$

$$= \frac{d}{dx}\{\beta \cos 2s\pi + (-)^{\frac{1}{2}} \beta \sin 2s\pi\}$$

$$= \frac{d}{dx}(\beta \cos 2s\pi) + (-)^{\frac{1}{2}} \frac{d}{dx}(\beta \sin 2s\pi),$$

$$= \frac{d}{d\alpha}(\beta \cos 2s\pi) + (-)^{\frac{1}{2}} \frac{d}{d\alpha}(\beta \sin 2s\pi), \text{ since } x = \alpha.$$

But x', y', z', representing the quantitive co-ordinates of any point of the curve, we have by the paper in the last number of the Journal to which we have before referred, putting $r = 0$,

$$x' = \alpha, \quad y' = \beta \cos 2s\pi, \quad \text{and } z' = \beta \sin 2s\pi.$$

$$\text{Hence } \frac{dy}{dx} = \frac{dy'}{dx'} + (-)^{\frac{1}{2}} \frac{dz'}{dx'};$$

and therefore, since $x = \alpha = x'$,

$$\frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2} + (-)^{\frac{1}{2}} \frac{d^2z'}{dx'^2},$$

$$\frac{d^3y}{dx^3} = \frac{d^3y'}{dx'^3} + (-)^{\frac{1}{2}} \frac{d^3z'}{dx'^3},$$

&c. = &c.

Suppose now that τ of the coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ have values of the form $+^0\rho$; then clearly, from the relations which we have established, we must have

$$\frac{dz'}{dx'} = 0, \quad \frac{d^2z'}{dx'^2} = 0, \quad \frac{d^3z'}{dx'^3} = 0, \ \ldots\ldots \ \frac{d^\tau z'}{dx'^\tau} = 0,$$

or the 'x curves' which pass through the point $x = a$, $y = ca^3$, corresponding to a regulative coefficient $+^0$, have a contact of the r^{th} order with the plane of x', y', or, which is the same thing, with the plane $(+, +)$.

Several writers on the Calculus have erroneously supposed that $\frac{dy}{dx}$, at a conjugate point, is necessarily of the form

$$\rho(\cos\theta + -^{\frac{1}{2}}\sin\theta),$$

sin θ being considered a finite quantity.
See, for instance, Hall's *Differential and Integral Calculus*, a work generally used in this University.

9. It may be well to take the present opportunity to offer a few observations on a theory which has been frequently proposed on the subject of conjugate points; the theory is, that they are in all cases vanishing ovals. See the *Encyclopédie Méthodique*, under the word *Conjuguée;* or see Montucla's *Histoire des Mathematiques*. This theory has been adopted by Professor De Morgan, in his Treatise on the Differential and Integral Calculus in the *Library of Useful Knowledge;* whose remarks on this subject we will present in his own words:—"I call the conjugate point an *evanescent oval*, because it never exists except where the equation is a degenerate variety of a wider class, each curve of which has an oval. The most simple case is that of $(x-a)^2 + (y-b)^2 = 0$, which belongs to no point except (a, b). This conjugate point is the circle described with a radius $= 0$, or an evanescent circle. Again,

$$y = \pm\sqrt{\{x(x-a)(x-b)\}},$$

a and b being positive, and $b > a$, consists of an oval from $x = 0$ to $x = a$, an unoccupied interval from $x = a$ to $x = b$, and infinite branches above and below this axis from $x = b$ upwards. As a diminishes, the oval becomes smaller, and finally when $a = 0$, the form of the equation becomes $y = x\sqrt{x-b}$, which gives $y = 0$ when $x = 0$, or the origin is a point of the curve: but there is no further point until $x = b$." Now the number of tangents which can be drawn to a finite oval is evidently unlimited, and consequently the value of $\frac{dy}{dx}$, when the oval vanishes and all its constituent points coalesce, must remain perfectly indeterminate. Thus, if we take the equation

$$(x-a)^2 + (y-b)^2 = r^2,$$

we have by differentiation

$$x - a + (y-b)\frac{dy}{dx} = 0,$$

$$\text{and therefore } \frac{dy}{dx} = -\frac{x-a}{y-b}.$$

Suppose now that r becomes zero; then the equation to the curve will become

$$(x-a)^2 + (y-b)^2 = 0.$$

But x and y being confined to quantitive values, must evidently in this case be equal to a and b respectively; and the equation is satisfied thus without the existence of any determinate vanishing ratio between $x - a$ and $y - b$, and thus $\frac{dy}{dx} = \frac{0}{0}$, the $\frac{0}{0}$ being perfectly indeterminate. According to the theory then of the vanishing oval, the test of the existence of a conjugate point would be the essential indeterminateness of $\frac{dy}{dx}$.

According to the theory of conjugate points which I have laid down, the evanescence of an oval is merely accessary to the existence of the point. In fact, when the oval vanishes the branches of the curve which deviate from the plane $+$, $+$, at its four extremities, corresponding to the minimum and maximum values of x and y, as we have explained in sect. 6, all come to have a common point, and thus constitute the complex multiplicity which characterizes a conjugate point. The advantage of the theory which I have expounded, in comparison with that which has just been described, consists in its more natural connection with the system of interpretation of geometrical equations, which exhibits an entire correspondence between the degree of the equation in x and y and the geometrical characters of its locus.

10. Suppose that when x is put equal to a in the equation (I), two of the 'x curves' meet each other in the plane $(+, +)$, at a point where y is equal to b: suppose also that when $x = a - h$, where h denotes any small magnitude affected by $+$ or $-$, these two curves lie without the plane $(+, +)$, and when $x = a + h$, lie within it. Then clearly, since roots of the form $k + (-)^{\frac{1}{2}} l$, which belong to the equation $f\{(a-h),\ y\} = 0$, must by pairs degenerate into roots of the form $+^0\rho$, when $a + h$ replaces $a - h$, it is clear that the equations to the two 'x curves', which at present occupy our attention, must be susceptible of expression under the form

$$y = \phi(x) \pm (x - a)^{\frac{2p+1}{2q}}\ \psi(x),$$

where p and q denote integral numbers.

Hence we have

$$\frac{dy}{dx} = \frac{d\phi(x)}{dx} \pm \frac{2p+1}{2q}(x-a)^{\frac{2p+1}{2q}-1}.\psi(x) \pm (x-a)^{\frac{2p+1}{2q}}.\frac{d\psi(x)}{dx}.$$

In this expression put $x = a$; and then we get

$$\frac{dy}{dx} = \frac{d\phi(a)}{da}, \text{ or } = \pm\,\infty,$$

according as $\frac{2p+1}{2q}$ is greater or less than unity.

The latter of these values for $\frac{dy}{dx}$ shews that the two curves are

continuously connected in the plane $+, +$, at the point $x=a$, $y=b$, and tend in opposite directions; that is, it corresponds to a maximum or minimum value of x in the plane $+, +$, and belongs to the case which we have discussed in section 4. The former of the values for $\frac{dy}{dx}$ shews that the tangents in the plane $+, +$, at points in the two 'x curves', coalesce at the point $x=a$, $y=b$, and corresponds to what is called a cusp; it being evident from this that there can be no such a thing as a cusp with a finite angle.

If the coalescent tangents of a cusp be oblique to the axes of x and y, it is easily seen that for any value of x the equation (I) gives two values for y, generally different, but coincident at the cusp, and *vice versâ*, interchanging x and y. Hence, at a cusp under these circumstances, we must have

$$\frac{d}{da}f(a, b) = 0, \quad \frac{d}{db}f(a, b) = 0.$$

Secondly, let the coalescent tangents be parallel to the axis of x. Then clearly for any assignable value of x we shall have from the equation (I), two values of y, generally different, but coinciding at the cusp.

Hence we have necessarily

$$\frac{d}{db}f(a, b) = 0.$$

But by differentiating the equation (I) with respect to x, and putting $x=a$, $y=b$, in the result, we have

$$\frac{d}{da}f(a, b) + \frac{d}{db}f(a, b)\frac{db}{da} = 0;$$

and since $\frac{db}{da}$ must be equal to zero, we have

$$\frac{d}{db}f(a, b) \cdot \frac{db}{da} = 0,$$

and therefore

$$\frac{d}{da}f(a, b) = 0;$$

and similarly if the coalescent tangents had been parallel to the axis of y we should have proved these two relations. Hence generally we see, that the conditions sufficient for the existence of complex multiplicity are always satisfied at a cusp, which is therefore a complex multiple point where in one direction the branches lie within the plane $+, +$, and in the other deviate from it. The multiplicity at a cusp will clearly always be even, both for the 'x' and for the 'y curves', except when $\frac{dx}{dy}$ or $\frac{dy}{dx}$ respectively are equal to zero, and the cusp is of the first species.

The algebraical problem of a plurality of branches at a point in the plane $+, +,$ was first noticed and discussed by Rolle in the *Journal des Sçavans* for the year 1702, under the title of 'Regles et Remarques pour le Probleme general des Tangents,' by purely algebraical considerations, where it was appealed to by him as an instance of the insufficiency of the differential calculus. Saurin replied to Rolle's memoir in the same Journal for the year 1703, shewing that the principles developed by De l'Hopital in l'Anal. des Infin. Pet., were perfectly adequate to a solution of the problem. There are papers on the same subject by Saurin also in the *Memoires de l'Academie* for 1716 and 1723. The whole theory was presented under a more regular aspect by Camus, in the *Memoires de l'Academie* for 1747.

11. It may evidently happen sometimes, that at one and the same point in the plane $+, +,$ there may coexist a conjugate point, a cusp, a point of simple multiplicity, and an ordinary multiple point. Under these circumstances it is plain that the order of the multiplicities of the 'x' and of the 'y curves' will be such as to comprehend the multiplicity due to each of the four coincident points; and will be represented by the number of the relations which we have exhibited in section (4).

IV.—INVESTIGATION OF SERIES FOR THE APPROXIMATE VALUES OF DEFINITE INTEGRALS.

The method of the separation of symbols may be conveniently applied to the investigation of the series for the approximate values of Definite Integrals, in terms of small increments of the variables.

Let a and b be the extreme values of the variable, and let the interval $a - b$ be divided into n parts, each equal to h, so that $a - b = nh$. Then if, as has been done in former articles, we put

$$\varepsilon^{\frac{d}{dx}} = \mathrm{D},$$

$$f(x + nh) - f(x) = (\mathrm{D}^{nh} - 1) f(x) \ \ldots\ldots\ldots \ (1).$$

Now $$\mathrm{D}^{nh} - 1 = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1}(\mathrm{D}^{h} - 1) = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1}\left(\varepsilon^{h\frac{d}{dx}} - 1\right)$$

$$= \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1}\left(h\frac{d}{dx} + \frac{h^2}{1.2}\frac{d^2}{dx^2} + \frac{h^3}{1.2.3}\frac{d^3}{dx^3} + \text{\&c.}\right)$$

Substituting this value in (1), and integrating and representing

$$\int f(x)\, dx \text{ by } f_1(x),$$

we have

$$f_{,}(x + nh) - f_{,}(x) = \frac{\mathrm{D}^{nh} - 1}{\mathrm{D}^{h} - 1}\left\{h f(x) + \frac{h^2}{1.2} f'(x) + \frac{h^3}{1.2} f''(x) + \text{\&c.}\right\}$$

But $\dfrac{D^{nh}-1}{D^{h}-1} = D^{(n-1)h} + D^{(n-2)h} + \&c. + D^{h} + 1.$

Substituting this expression written in the contrary order,

$$f_{,}(x+nh) - f_{,}(x) =$$

$$= (1+D^{h}+\&c.+D^{(n-1)h})\,\{hf(x)+\frac{h^2}{1.2}f'(x)+\&c.\}$$

$$= h\,[f(x)+f(x+h)+\&c.+f\{x+(n-1)h\}]$$

$$+\frac{h^2}{1.2}[f'(x)+f'(x+h)+\&c.+f'\{x+(n-1)h\}]$$

$$+\ \&c.$$

Now, putting $x = a$, and consequently $x + nh = b$, we have

$$f_{,}(b)-f_{,}(a)=h\,[f(a)+f(a+h)+\&c.+f\{a+(n-1)h\}]$$

$$+\frac{h^2}{1.2}[f'(a)+f'(a+h)+\&c.+f'\{a+(n-1)h\}]$$

$$+\ \&c.$$

Again, since we have

$$1+D^{h}+\&c.+D^{(n-1)h}=\frac{D^{nh}-1}{D^{h}-1}=(D^{nh}-1)\,(\epsilon^{h\frac{d}{dx}}-1)^{-1},$$

we may expand the factor $(\epsilon^{h\frac{d}{dx}}-1)^{-1}$ by Bernoulli's numbers, when it becomes

$$\frac{1}{h}\left(\frac{d}{dx}\right)^{-1} - \frac{1}{2} + \frac{B_1}{1.2}h\frac{d}{dx} - \frac{B_3}{1.2.3.4}h^3\frac{d^3}{dx^3} + \&c.$$

We therefore have

$$(1+D^{h}+D^{2h}+\&c.+D^{(n-1)h})f(x)$$

$$=(D^{nh}-1)\left\{\frac{1}{h}\left(\frac{d}{dx}\right)^{-1}-\frac{1}{2}+\frac{B_1}{1.2}h\frac{d}{dx}-\frac{B_3}{1.2.3.4}h^3\left(\frac{d}{dx}\right)^3+\&c.\right\}f(x)$$

$$=(D^{nh}-1)\left\{\frac{1}{h}f_{,}(x)-\frac{1}{2}f(x)+\frac{B_1}{1.2}hf'(x)-\frac{B_3}{1.2.3.4}h^3f'''(x)+\&c.\right\}$$

Whence, by transposition and multiplication by h,

$$(D^{nh}-1)f_{,}(x)=h\,[\tfrac{1}{2}f(x)+f(x+h)+\&c.+f\{x+(n-1)h\}$$
$$+\tfrac{1}{2}f(x+nh)]$$

$$-(D^{nh}-1)\frac{B_1}{1.2}h^2f'(x)$$

$$+(D^{nh}-1)\frac{B_3}{1.2.3.4}h^4f'''(x)$$

$$-\ \&c.$$

Now substituting a for x and b for $x+nh$, we have

$$f_{\prime}(b)-f_{\prime}(a)=h[\tfrac{1}{2}f(a)+f(a+h)+\&c.+f\{a+(n-1)h\}+\tfrac{1}{2}f(b)]$$
$$-\frac{B_1}{1.2}h^2\{f'(b)-f'(a)\}$$
$$+\frac{B_3}{1.2.3.4}h^4\{f'''(b)-f'''(a)\}$$
$$-\&c.$$

See Poisson *Mécanique*, Vol. I., p. 23.

H. T.

V.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By R. L. Ellis, B.A., Fellow of Trinity College.

It is shown in the theory of the earth's figure, that if the pressure and density at any point be connected by the equation

$$dp = k\rho\, d\rho,$$

where k is a constant, then the ellipticity of the surface may be deduced from the solution of the equation

$$\frac{d^2y}{dx^2} + q^2y = \frac{6y}{x^2}.$$

This equation is not easily integrated. La Place, in the eleventh book of the *Mécanique Céleste*, (v. 51), gives a solution of it, but without demonstration; and the lacuna thus left is not supplied in the works on the subject generally made use of in Cambridge.

Mr. Gaskin has however effected the integration of

$$\frac{d^2y}{dx^2} + q^2y = \frac{p(p-1)}{x^2}y,$$

when p is integral, in finite terms, (vide Hymers' *Diff. Eq.* p. 53), and the proposed equation is a case of this one. But perhaps a more direct analysis is preferable, as it enables us to extend our method to two or three classes of equations of all orders. One of these will be considered in the present paper—another, the solution of which admits of a remarkable symbolical form, will be given in the next number of the *Journal.*

We shall begin with the particular equation which occurs in the theory of the earth's figure, both because from its physical application it has an interest for some who care but little for pure analysis, and because it will exemplify the general method.

$$\frac{d^2y}{dx^2} + q^2y = 6\frac{y}{x^2} \;\ldots\ldots\;.\;\ldots\ldots\; (1).$$

Let $$y = \Sigma a_n x^n \text{ ....}(2),$$

$$\therefore \ \{n(n-1)-6\} \ a_n + q^2 a_{n-2} = 0 \ldots(3),$$

$$n(n-1)-6 = n(n-1)-3(3-1) = (n-3)(n+2),$$

$$\therefore \ (n-3)(n+2) a_n + q^2 a_{n-2} = 0 \ldots (4).$$

To get rid of the factor $(n-3)$, assume

$$(n+2) a_n = (n-1) b_n \text{ ...... } (5),$$

$$\therefore \ a_{n-2} = \frac{n-3}{n} b_{n-2},$$

$$\therefore \ n(n \quad 1) b_n + q^2 b_{n-2} = 0 \ldots (6).$$

Hence b_n is made to depend on b_1 or b_0 as n is odd or even, and we see at once that

$$\Sigma b_n x^n = b_0 \cos qx + b_1 \sin qx,$$

or changing the constants,

$$\Sigma b_n x^n = \text{C} (\sin q\, x + \alpha) \text{......} (7).$$

Also by (5),

$$a_n = b_n - 3 \frac{b_n}{n+2} = b_n + \frac{3}{q^2}(n+1) b_{n+2} \text{ ..}(8),$$

for by (6),

$$(n+1) b_{n+2} + \frac{q^2}{n+2} b_n = 0,$$

$$\therefore \ \Sigma a_n x^n = \Sigma b_n x^n + \frac{3}{q^2} \Sigma (n-1) b_n x^{n-2}$$

$$= \Sigma b_n x^n + \frac{3}{q^2} \left(\frac{\Sigma b_n x^n}{x}\right)',$$

$$\therefore \ y = \text{C} \sin (qx + \alpha) + \frac{3\text{C}}{q^2} \left(\frac{\sin (qx+\alpha)}{x}\right)' \text{ ...... } (9),$$

the complete solution, which may be written thus,

$$y = \text{C} \left\{ \sin (qx+\alpha) \left(1 - \frac{3}{q^2 x^2}\right) + \frac{3}{qx} \cos (qx + \alpha) \right\} \ldots (10).$$

We now proceed to the more general equation,

$$\frac{d^2 y}{dx^2} + q^2 y = p(p-1) \frac{y}{x^2} \text{ .... } (11).$$

As before, we shall get

$$\{n(n-1) - p(p-1)\} \ a_n + q^2 a_{n-2} = 0 \text{ ..... } (12).$$

Now $$n(n-1) - p(p-1) = (n-p)(n+p-1) \ldots (13),$$

which is the fundamental principle of our analysis,

$$\therefore \ (n-p)(n+p-1) \ a_n + q^2 a_{n-2} = 0 \text{ ...... } (14).$$

Assume $$(n+p-1) \ a_n = (n-p+2) \ b_n \text{ .... } (15),$$

$$\therefore \ a_{n-2} = \frac{n-p}{n+p-3} b_{n-2}$$

and $$(n-p+2)(n+p-3) \ b_n + q^2 b_{n-2} = 0 \ldots (16).$$

Again, assume

$$(n+p-3)\,b_n = (n-p+4)\,c_n \;\ldots\ldots\ldots (17),$$

and so on successively. Thus we shall get a series of equations, of which

$$(n-p+\mu)\,(n+p-\mu-1)\,l_n + q^2 l_{n-2} = 0 \ldots\ldots (18),$$

is the general type, where μ is even.

If p is even, let $p=\mu$, $\therefore\; p-\mu-1=-1$.

If it is odd, let $p=\mu+1$, $\therefore\; -p+\mu=-1$:

and in both cases (15) becomes

$$n\,(n-1)\,l_n + q^2 l_{n-2} = 0,$$

and therefore

$$\Sigma l_n x^n = \mathrm{C}\sin(qx+a) \ldots\ldots (19).$$

Let $\{n-p+(\mu-2)\}\,(n+p-\mu+1)\,i_n + q^2 i_{n-2} = 0,$

$$(n-p+\mu)\,(n+p-\mu-1)\,k_n + q^2 k_{n-2} = 0,$$

be any two consecutive equations; then

$$(n+p-\mu+1)\,i_n = (n-p+\mu)\,k_n \ldots\ldots\ldots (20),$$

$$n-p+\mu = n+p-\mu+1-2\,(p-\mu)-1,$$

$$\therefore\; i_n = k_n - \{2\,(p-\mu)+1\}\,\frac{k_n}{n+p-\mu+1} \ldots (21),$$

and $$\frac{k_n}{n+p-\mu+1} = -\frac{1}{q^2}(n+2-p+\mu)\,k_{n+2},$$

$$\therefore\; \Sigma i_n x^n = \Sigma k_n x^n + \frac{2\,(p-\mu)+1}{q^2}\,\Sigma(n-p+\mu)\,k_n x^{n-2}.$$

Now $(n-p+\mu)\,k_n x^{n-2} = x^{p-\mu-1}\,(n-p+\mu)\,k_n x^{n-p+\mu-1}$

$$= x^{p-\mu-1}\left(\frac{k_n x^n}{x^{p-\mu}}\right)',$$

$$\therefore\; \Sigma i_n x^n = \Sigma k_n x^n + \frac{2\,(p-\mu)+1}{q^2}\,x^{p-\mu-1}\left(\frac{\Sigma k_n x^n}{x^{p-\mu}}\right)' \ldots (22).$$

By the application of this formula, y or $\Sigma a_n x^n$ may be deduced by a series of regular operations from C sin $(qx+a)$.

If p is even, $2\,(p-\mu)+1$ gives the series 1, 5, 9, &c.

If it is odd, the series is 3, 7, 11, &c.

Particular cases may be solved by (22) with considerable facility. By inspection we have

$$y = \mathrm{C}\left\{\sin(qx+a) + \frac{1}{qx}\cos(qx+a)\right\}$$

for the solution of

$$\frac{d^2y}{dx^2} + q^2 y = \frac{2y}{x^2}.$$

The solution of $\frac{d^2y}{dx^2} + q^2y = \frac{20y}{x^2}$, where $p = 5$, is easily seen to be

$$y = \mathrm{C}\left\{\sin(qx + a) + \frac{3}{q^2}\left(\frac{\sin(qx+a)}{x}\right)'\right\}$$
$$+ \frac{7\mathrm{C}}{q^2}\, x^2\left[\frac{1}{x^3}\left\{\sin(qx + a) + \frac{3}{q^2}\left(\frac{\sin(qx+a)}{x}\right)\right\}\right]'.$$

Lastly, the solution of

$$\frac{d^2y}{dx^2} + q^2y = \frac{12y}{x^2},$$

is $y = \mathrm{C}\left(\sin(qx + a) + \frac{1}{qx}\cos(qx + a)\right)$

$$+ \frac{5\mathrm{C}}{q^2}\, x\left\{\frac{1}{x^2}\left(\sin(qx + a) + \frac{1}{qx}\cos(qx + a)\right)\right\}'.$$

The second line is equivalent to

$$\frac{5\mathrm{C}}{q^2x}\left(q\cos(qx+a) - \frac{1}{qx^2}\cos(qx+a) - \frac{1}{x}\sin(qx+a)\right.$$
$$\left. - \frac{2}{x}\sin(qx+a) - \frac{2}{qx^2}\cos(qx+a)\right),$$

and thus

$$y = \mathrm{C}\left(\sin(qx+a) + \frac{6}{qx}\cos(qx+a) - \frac{15}{q^2x^2}\sin(qx+a)\right.$$
$$\left. - \frac{15}{q^3x^3}\cos(qx+a)\right).$$

These examples will sufficiently illustrate the general formula.

The same method is applicable to the equation

$$\frac{d^3y}{dx^3} + q^3y = p(p-1)\frac{1}{x^2}\frac{dy}{dx} \;\ldots\ldots\; (23).$$

Here we have

$$n\{(n-1)(n-2) - p(p-1)\}\, a_n + q^3a_{n-3} = 0,$$
$$\therefore\; n(n-p-1)(n+p-2)\, a_n + q^3a_{n-3} = 0.$$

Let $\qquad (n+p-2)\, a_n = (n-p-1+3)\, b_n,$

$$\therefore\; n(n-p-1+3)(n+p-2-3)\, b_n + q^3b_{n-3} = 0;$$

and generally

$$n(n-p-1+\nu)(n+p-2-\nu)\, l_n + q^3l_{n-3} = 0 \;\ldots..\; (24),$$

where ν is divisible by 3.

If p is so too, let $p = \nu$,

$$\therefore\; n-p-1+\nu = n-1 \text{ and } n+p-2-\nu = n-2.$$

If $p - 1$ is divisible by 3, let $p - 1 = \nu$,

$$\therefore\; -p-1+\nu = -2 \text{ and } p-2-\nu = -1;$$

and in both cases (24) becomes

$$n(n-1)(n-2)\,l_n+q^3l_{n-3}=0,$$

and therefore $\Sigma l_n x^n$ fulfils the equation

$$\frac{d^3z}{dx^3}+q^3z=0 \quad \ldots\ldots\ldots\ldots (25).$$

Again, if

$$n(n-p-1+\nu-3)(n+p-2-\nu+3)\,l_n+q^3l_{n-3}=0,$$
$$n(n-p-1+\nu)(n+p-2-\nu)\,k_n+q^3k_{n-3}=0,$$

be any consecutive equations, we have

$$(n+p-2-\nu+3)\,i_n=(n-p-1+\nu)\,k_n,$$
$$\therefore\ i_n=k_n-2(p+1-\nu)\frac{k_n}{n+p+1-\nu}.$$

Also, $\dfrac{k_n}{n+p+1-\nu}=-\dfrac{1}{q^3}(n+3)\,(n+3)+p-1+\nu)\,k_{n+3}$,

$$\therefore\ \Sigma i_nx^n=\Sigma k_nx^n+\frac{2(p+1-\nu)}{q^3}\Sigma n(n-p-1+\nu)\,k_nx^{n-3},$$

$$\text{or } \Sigma i_nx^n=\Sigma k_nx^n+\frac{2(p+1-\nu)}{q^3}x^{p-\nu-1}\left(\frac{(\Sigma k_nx^n)'}{x^{p-\nu}}\right)' \ldots (26),$$

as may easily be seen *à priori*, or verified by differentiation. The formula (26) is used in the same way as (22), to which it is analogous.

We will give one instance of its application,

$$\frac{d^3y}{dx^3}+q^3y=\frac{6}{x^2}\frac{dy}{dx}.$$

The solution of

$$\frac{d^3z}{dx^3}+q^3z=0, \text{ is}$$

$$z=C_1e^{-qx}+C_2e^{\frac{q}{2}x}\sin\left(\frac{\sqrt{3}}{2}qx+a\right),$$

and by (26),

$$y=z+\frac{2.(3+1-3)}{q^3}x^{-1}\left(\frac{dz}{x^0dx}\right)',$$

$$\text{or } y=z+\frac{2}{q^3x}\frac{d^2z}{d^2x}, \text{ which gives}$$

$$y=C_1e^{-qx}\left(1+\frac{2}{qx}\right)+$$
$$+C_2e^{\frac{q}{2}x}\left\{\sin\left(\frac{\sqrt{3}}{2}qx+a\right)\left(1-\frac{1}{qx}\right)+\frac{\sqrt{3}}{qx}\cos\left(\frac{\sqrt{3}}{2}qx+a\right)\right\},$$

for the complete solution of the proposed equation.

It is obvious that analogous equations exist in all orders, and that when p is of certain forms,

$$\frac{d^m y}{dx^m} + q^m y = p(p-1)\frac{1}{x^2}\frac{d^{m-2}}{dx^{m-2}}y$$

may be integrated in finite terms.

It will be sufficient, after what has been said for the cases of $m = 2$ and $= 3$, to state the results of the general investigation; they may be very readily deduced by the same method as that we have already used.

The process succeeds when either of the factors p or $p - 1$ is divisible by m, and the general formula of which (22) and (26) are cases, is

$$\Sigma i_n x^n = \Sigma k_n x^n + \frac{2(p-rm)+m-1}{q^m} x^{p-rm-1}\left(x^{-(p-rm)}\frac{d^{m-2}}{dx^{m-2}}\Sigma k_n x^n\right)' \quad \ldots\ldots (27).$$

Particular cases may however be easily solved without reference to this formula; thus, if we had

$$\frac{d^4 y}{dx^4} - q^4 y = \frac{12}{x^2}\frac{d^2 y}{dx^2},$$

we should proceed as follows:

$$n(n-1)\{(n-2)(n-3)-4\cdot 3\}\, a_n - q^4 a_{n-4} = 0,$$

$$n(n-1)(n-6)(n+1)\, a_n - q^4 a_{n-4} = 0,$$

$$(n+1)\, a_n = (n-2)\, b_n,$$

$$\therefore\ n(n-1)(n-2)(n-3)\, b_n - q^4 b_{n-4} = 0,$$

$$\therefore\ (\Sigma b_n x^n)^{iv} - q^4 \Sigma b_n x^n = 0,$$

$$\therefore\ \Sigma b_n x^n = C_1 e^{qx} + C_2 e^{-qx} + C_3 \sin(qx+a),$$

$$\text{and } a_n = b_n - \frac{3}{n+1} b_n = b_n - \frac{3}{q^4}(n+4)(n+3)(n+2)\, b_{n-4},$$

$$\therefore\ y = \Sigma b_n x^n - \frac{3}{q^4}\Sigma n(n-1)(n-2)\, b_n x^{n-4},$$

$$= \Sigma b_n x^n - \frac{3}{q^4 x}\frac{d^3}{dx^3}\Sigma b_n x^n,$$

$$\therefore\ y = C_1 e^{qx}\left(1 - \frac{3}{qx}\right) + C_2 e^{-qx}\left(1 + \frac{3}{qx}\right)$$

$$+ C_3\left(\sin(qx+a) + \frac{3}{qx}\cos(qx+a)\right).$$

The principle of our analysis, it has already been remarked, is contained in the equation

$$n(n-1) - p(p-1) = (n-p)(n+p-1);$$

and this consideration suggests an extension of it.

For it is obvious that the coefficients of a_n in

$$\frac{d^m y}{dx^m}, \text{ and in } p\,(p-1)\,\frac{d^s}{dx^s}\left(\frac{1}{x^2}\,\frac{d^{m-s-2}}{dx^{m-s-2}}\,y\right)$$

differ only in this, that where the first has the factors

$$(n-m+s+2)\,(n-m+s+1),$$

the second has $p\,(p-1)$.

Thus the same transformation applies; and if either p, or $p-1$ is divisible by m, the solution of

$$\frac{d^m y}{dx^m}+q^m y=p\,(p-1)\,\frac{d^s}{dx^s}\left(\frac{1}{x^2}\,\frac{d^{m-s-2}}{dx^{m-s-2}}\,y\right) \ldots\ldots (28),$$

may be made to depend on that of

$$\frac{d^m y}{dx^m}+q^m y=0,$$

and thus effected in finite terms.

The formula of reduction in this case is a little more complicated than those already given, and we will not dwell longer upon it, our object being rather to point out the integrability of certain classes of equations than actually to integrate them.

The equation

$$n\,(n-1)-p\,(p-1)=(n-p)\,(n+p-1),$$

is a particular case of

$$n\,(n-\mu)-p\,(p-\mu)=(n-p)\,(n+p-\mu),$$

and the latter will give us various formulæ of reduction according to the value of μ. Thus

$$\frac{d^2 y}{dx^2}-\frac{1}{x}\,\frac{dy}{dx}+q^2 y=p\,(p-2)\,\frac{y}{x^2},$$

may be reduced to

$$\frac{d^2 y}{dx^2}-\frac{1}{x}\,\frac{dy}{dx}+q^2 y=0,$$

for the coefficient of a_n in the former is

$$n\,(n-2)-p\,(p-2)=(n-p)\,(n+p-2),$$

which, provided p is even, may be reduced to $n\,(n-2)$. But in this, and in analogous cases, the auxiliary equation is, apparently, insoluble.

The applicability of our transformation would, it is evident, not be affected, if the equation were, instead of (25),

$$\frac{d^m y}{dx^m}+q^{m-t}\,\frac{d^t y}{dx^t}=0, \text{ \&c.} \ldots\ldots\ldots (29);$$

and, provided p or $p-1$ were divisible by $m-t$, (29) might be reduced to

$$\frac{d^m y}{dx^m}+q^{m-t}\,\frac{d^t y}{dx^t}=0.$$

But this case requires more care than those already considered, as if certain factors which apparently disappear are neglected, our solution is incomplete, or erroneous.

An instance will make this clear,

$$\frac{d^2y}{dx^2} + q\frac{dy}{dx} = 2\frac{y}{x^2}.$$

Here $\quad (n-2)(n+1)a_n + (n-1)q\, a_{n-1} = 0 \ldots\ldots(A).$

Let $\quad (n+1)a_n = (n-1)b_n \ldots\ldots\ldots\ldots(a),$

$\therefore \; (n-2)(n-1)n\, b_n + (n-2)(n-1)q\, b_{n-1} = 0\ldots(B).$

The factor $n-2$ may be safely neglected. But $n-1$ is essential, because the solution of the auxiliary equation

$$\frac{d^2z}{dx^2} + q\frac{dz}{dx} = 0,$$

gives $\quad (n-1)(n b_n + q\, b_{n-1}) = 0,$

and would be incomplete if we omitted the first factor.

From (a) we get

$$a_n = b_n - 2\frac{b_n}{n+1},$$

and, as except when $n = 1$, there is

$$n\, b_n + q\, b_{n-1} = 0, \quad \frac{b_n}{n+1} = -\frac{1}{q}b_{n+1}, \quad \text{unless } n = 0,$$

$$\therefore \; a_n = b_n + \frac{2}{q}b_{n+1} \ldots\ldots\ldots (a').$$

Now the solution of the auxiliary equation is

$$z = c_1 + c_2 e^{-qx};$$

and from (a') we deduce

$$y = z + \frac{2}{qx}z;$$

$$\text{therefore } \; y = \left(1 + \frac{2}{qx}\right)(c_1 + c_2 e^{-qx})$$

is apparently the solution of the proposed equation. But it will be found not to satisfy it, unless $c_1 = 0$, and then

$$y = c_2\left(1 + \frac{2}{qx}\right)e^{-qx}$$

is only a particular solution. The reason is, that in laying down (a') as generally true, we imply that

$$n\, b_n + q\, b_{n-1} = 0$$

is true for $n = 1$; whereas the equation which contains the solution of $\dfrac{d^2z}{dx^2} + q\dfrac{dz}{dx} = 0$, viz.

$$(n-1)(n\, b_n + q\, b_{n-1}) = 0,$$

shows that b_1 is not *necessarily* connected with b_0; and that if we

assume such connection, we get only a particular solution. Hence our formula of reduction implies the connection of b_1 and b_0; while their independence is implied in the general solution of the auxiliary equation, to which this formula is applied; and these contradictory suppositions lead to an erroneous result. To put $c_1 = 0$, is to connect b_0 and b_1, or, which is the same thing, to neglect the factor $n - 1$; and the value of y thus got is therefore a solution, but not the complete solution of the proposed equation.

To complete it, we must, bearing in mind the independence of b_0, recur to (a), which is always true,

$$\therefore\ a_0 = -b_0;$$

and from (a'), which is true for $n = -1$, we get

$$a_{-1} = b_{-1} + \frac{2}{q} b_0.$$

Now b_{-1} is obviously $= 0$,

$$\therefore\ a_{-1} = -\frac{2}{q} a_0;$$

and these two quantities are independent of a_1, a_2, &c.,

$$\therefore\ y = a_0 \left(1 - \frac{2}{qx}\right)$$

is a particular solution, and

$$y = c_1 \left(1 - \frac{2}{qx}\right) + c_2 \left(1 + \frac{2}{qx}\right) e^{-qx}$$

is the complete solution of the proposed equation.

The method of proceeding suggested by this example, is to obtain a solution, neglecting all factors analogous to $(n-1)$, and then to complete it by reference to the assumptions of transformation, such as (a), which have been made use of.

The equations which we have solved are not a very numerous nor perhaps an important class. But one of them, at least, is susceptible of a physical application of great interest; and so few equations of the higher orders are integrable in finite terms, that the discussion of those which are, has always some degree of value.

VI.—ANALYTICAL SOLUTIONS OF PROBLEMS IN PLANE ASTRONOMY.

To the Editor of the Cambridge Mathematical Journal.

SIR,—In the fourth number of your Journal, you inserted the analytical solution of certain problems of Plane Astronomy. The following are other problems belonging to the same subject, of which the analytical solutions are extremely simple. Perhaps they may interest some of your readers.

1. To find how much the time of a star's rising is altered by refraction,—(Maddy's *Astronomy*, p. 176.)

Let r denote the refraction.

The problem consists in finding the difference between the hour-angle of the star corresponding to the zenith distance 90°, and that corresponding to the zenith distance $90° + r$. This difference divided by 15° will give the difference between the star's apparent and real time of rising.

Now, if z denote the zenith distance of a heavenly body, δ its declination, h its hour-angle, l the latitude of the place,

$$\cos z = \sin l \sin \delta + \cos h \cos l \cos \delta.$$

Therefore if Δh denote the small difference of h corresponding to the difference Δz of z,

$$\sin z \, \Delta z = \sin h \cos l \cos \delta \, \Delta h,$$

$$\therefore \Delta h = \frac{\sin z \, \Delta z}{\sin h \cos l \cos \delta} \; \ldots\ldots \; (1).$$

In the actual problem $z = 90°$, and $\Delta z = r$;

$$\therefore \Delta h = \frac{r}{\sin h \cos l \cos \delta}.$$

But, since $z = 90°$, $\cos h = - \tan l \tan \delta$.

$$\therefore \sin h = \frac{\sqrt{\{(\cos l)^2 (\cos \delta)^2 - (\sin l)^2 (\sin \delta)^2\}}}{\cos l \cos \delta}$$

$$= \frac{\sqrt{\{\cos (l + \delta) \cos (l - \delta)\}}}{\cos l \cos \delta},$$

$$\therefore \Delta h = \frac{r}{\sqrt{\{\cos (l + \delta) \cos (l - \delta)\}}}.$$

2. To find the retardation of the moon's rising on successive days, in consequence of her motion in her orbit. (Maddy's *Astronomy*, p. 290.)

Let m represent the moon's daily motion in her orbit, n the inclination of her orbit to the horizon. Then $m \sin n$ is the moon's daily motion perpendicular to the horizon, and is therefore the

angle by which she is depressed below the horizon at the time when if she had no motion in her orbit she would be rising.

The problem consists, then, in finding the time in which the revolution of the earth about its axis changes the moon's zenith distance from $90^\circ + m \sin n$ to 90°. Consequently the calculations of this problem are precisely the same as those of the last. Equation (1) gives us, in this case,

$$\Delta h = \frac{m \sin n}{\sin h \cos l \cos \delta},$$
$$= \frac{m \sin n}{\sqrt{\{\cos (l + \delta) \cos (l - \delta)\}}}.$$

This quantity divided by 15 gives the retardation of the moon's rising in consequence of her motion in her orbit.

I am, Sir, &c.

A. C.

November 6th, 1840.

VII.—ANALYTICAL GEOMETRY.

By G. Boole, Lincoln.

1. In the miscellaneous investigations of the following paper, I shall have frequent occasion to refer to a theorem noticed by a writer in the first volume of this Journal, and which it may be as well here to recapitulate, viz.,

$$\begin{aligned}(ay-bx)^2+(bz-cy)^2+(cx-az)^2 &= (a^2+b^2+c^2)(x^2+y^2+z^2) \\ -(ax+by+cz)^2 &= (a^2+b^2+c^2)(x^2+y^2+z^2)\sin\theta^2\end{aligned} \quad \ldots(1),$$

where θ is the inclination of two lines, whose direction cosines are proportional to $a, b, c, x, y, z,$ respectively.

The functions $ay - bx$, $bz - cy$, $cx - az$, and others of similar form, are of very frequent occurrence in Analytical Mechanics and Geometry, and in questions generally, relative to space. They may in all cases be considered as resulting from an elimination between two linear equations. If those equations are of the form

$$lx + my + nz = 0,$$
$$l'x + m'y + n'z = 0,$$

which includes the conditions of perpendicularity for similar loci of the first degree, we obtain by successively eliminating the variables

$$\frac{x}{mn' - m'n} = \frac{y}{nl' - n'l} = \frac{z}{lm' - l'm};$$

whence it appears, that if $mn'-m'n$, $nl'-n'l$, $lm'-l'm$, are proportional to the direction cosines of a proposed straight line, that line will be perpendicular to the straight lines, and parallel to the planes whose direction cosines are l, m, n and l', m', n'. The converse of this proposition is true, when we take $mn'-m'n$, &c. proportional to the direction cosines of a proposed plane.

2. We shall next proceed to investigate a general expression for the minimum distance of two loci of the first degree; to accomplish which it will be sufficient, as might easily be shewn, to discuss generally the case of two straight lines.

If, according to the usual method, we seek the minimum value of the function

$$(x-x')^2+(y-y')^2+(z-z')^2,$$

the variables being connected by the equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \text{ .... } (2),$$

$$\frac{x'-a'}{l'}=\frac{y'-b'}{m'}=\frac{z'-c'}{n'} \text{ .... } (3),$$

we find after eliminating the differentials

$$l(x-x')+m(y-y')+n(z-z')=0 \text{......} (4),$$

$$l'(x-x')+m'(y-y')+n'(z-z')=0 \text{... ..} (5).$$

Multiply the successive terms of (4) by the corresponding terms of (2), and those of (5) by the corresponding terms of (3), and we obtain

$$(x-a)(x-x')+(y-b)(y-y')+(z-c)(z-z')=0,$$

$$(x'-a')(x-x')+(y'-b')(y-y')+(z'-c')(z-z')=0.$$

Whence, by subtraction and transposition,

$$(x-x')^2+(y-y')^2+(z-z')^2=(a-a')(x-x')+(b-b')(y-y')$$
$$+(c-c')(z-z'),$$

and dividing by $\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}$,

$$D=\frac{(a-a')(x-x')+(b-b')(y-y')+(c-c')(z-z')}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}},$$

or making $\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}=D$,

and $\sqrt{\{(a-a')^2+(b-b')^2+(c-c')^2\}}=D'$,

$$D=\frac{(a-a')(x-x')+(b-b')(y-y')+(c-c')(z-z')}{DD'}D';$$

whence, if ϕ be the inclination of the distance D' to the minimum distance D,

$$D=D'\cos\phi \text{ .... } (6).$$

3. Geometrical considerations alone make it evident, that this theorem holds true for the case of two parallel planes, and indeed for every possible case of minimum distance, between two given

loci whose analytical equations are of the first degree. To apply it to any particular case, it is only necessary to observe, from (4) and (5), that the line of minimum distance is always perpendicular to both the given loci.

Thus, to find the minimum distance of the planes, whose equations are

$$lx + my + nz = d \;\ldots\ldots\; (7),$$

$$l'x + m'y + n'z = d' \;\ldots\ldots\; (8),$$

we have, assuming a, b, c and a', b', c' as the co-ordinates of two points in (7) and (8),

$$D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}},$$

$$\cos\phi = \frac{l(a-a') + m(b-b') + n(c-c')}{\sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}},$$

whence $D = l(a-a') + m(b-b') + n(c-c') = d \sim d'$.

If we wish to obtain the distance of the point a', b', c', from the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \;\ldots\ldots\ldots\; (9),$$

we must express $\cos\phi$ by the sine of inclination of D' to the given line; thus we shall have

$$D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}},$$

$$\cos\phi = \sqrt{\left\{1 - \left(\frac{l(a-a') + m(b-b') + n(c-c')}{\sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}}\right)^2\right\}},$$

$$\therefore D = \sqrt{[(a-a')^2 + (b-b')^2 + (c-c')^2 - \{l(a-a') + m(b-b') + n(c-c')^2\}]}.$$

The value of D might, in this case, be put in a symmetrical form, by adopting the symmetrical expression for $\cos\phi$, instead of the one we have employed.

To express the condition that two lines may intersect or lie in the same plane, we have only to make the minimum distance equal to 0. Let the equations of the proposed lines be, as before,

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \;\ldots\ldots\ldots\; (10),$$

$$\frac{x'-a'}{l'} = \frac{y'-b'}{m'} = \frac{z'-c'}{n'} \;\ldots\ldots\ldots\; (11).$$

Here $D' = \sqrt{\{(a-a')^2 + (b-b')^2 + (c-c')^2\}}$,

$$\cos\phi = \frac{(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm)}{D'\sqrt{\{(mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2\}}}.$$

Hence the condition sought will be

$$(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm) = 0 \;\ldots\ldots (12).$$

This equation is similar in form to the one by which we express the condition, that these lines shall be parallel to or coincident with the same plane, viz.

$$(mn'-m'n)\,l''+(nl'-n'l)\,m''+(lm'-l'm)\,n''=0 \;..\; (13).$$

Lastly, to find the perpendicular from a given point a', b', c', on a proposed plane,

$$lx+my+nz=d,$$

we have, since the direction cosines of the line of minimum distance are l, m, n,

$$\cos\phi=\frac{l(a-a')+m(b-b')+n(c-c')}{\sqrt{\{(a-a')^2+(b-b')^2+(c-c')^2\}}};$$

whence
$$\begin{aligned}D&=l(a-a')+m(b-b')+n(c-c')\\&=la+mb+nc-d,\end{aligned}$$

a', b', c', being the co-ordinates of a point in the plane.

4. If we suppose the straight line defined by the equations of two intersecting planes, the investigation of minimum distances leads to some remarkable results. Let us, for example, seek the distance from the origin of co-ordinates, to the straight line formed by the intersection of the planes

$$lx+my+nz=d \;......\; (14),$$
$$l'x+m'y+n'z=d' \;......\; (15),$$

Here $D'=\sqrt{(x^2+y^2+z^2)}$

$$\cos\phi=\sqrt{\left\{1-\frac{\{x(mn'-m'n)+y(nl'-n'l)+z(lm'-l'm)\}^2}{(x^2+y^2+z^2)(\sin\theta)^2}\right\}},$$

where b equals the inclination of (14) and (15)

Wherefore, on reduction,

$$D=\frac{\sqrt{(x^2+y^2+z^2)(\sin\theta)^2-\{x(mn'-m'n)+y(nl'-n'l)+z(lm'-l'm)\}}}{\sin\theta} \;......\; (16).$$

Since the variables x, y, z, (according to the analogy of our former processes they ought to have been represented by a, b, c,) are only required to satisfy the conditions (14) and (15), we may in the value of D *arbitrarily* assume the value of any one of them, and determine the remaining two from those conditions. It follows from hence, that it must be *possible* to express the value of the second member of (16) by aid of (14) and (15), without any assumption whatever. This we are enabled to do by the theorem (1), after preparing our equations (14) and (15) by the successive elimination of the variables x, y, z. The result is,

$$D=\frac{\sqrt{(d^2-2dd'\cos\theta+d'^2)}}{\sin\theta} \;......\; (17).$$

θ being the inclination of d and d'.

If we observe that the extremities of the minimum distance, and

the intersections of the perpendiculars d and d' with the planes (14), (15), lie in the circumference of the same circle, the expression at which we have just arrived, will be seen to be equivalent to the geometrical theorem, that the ratio of any side of a plane triangle to the sine of its opposite angle, is equal to the diameter of the circumscribing circle.

5. The method by which we have arrived at the expression for D in the last problem is remarkable, and may be very advantageously employed in some cases of elimination. In particular it leads to an elegant and symmetrical solution of the system of equations,

$$lx + my + nz = d \text{ .... } (18),$$
$$l'x + m'y + n'z = d' \text{ .... } (19),$$
$$x^2 + y^2 + z^2 = 1 \text{ .... } (20),$$

subject, as we shall suppose, to the conditions

$$l^2 + m^2 + n^2 = 1,$$
$$l'^2 + m'^2 + n'^2 = 1.$$

Special cases of this system of equations are continually met with in Analytical Geometry. It is needless to observe, that the ordinary solution by quadratics is devoid of symmetry, and does not present any very obvious mode of reduction to a simpler form.

From (18) and (19) eliminate successively the variables, there results

$$(nl' - n'l)\,z - (lm' - l'm)\,y = dl' - d'l,$$
$$(lm' - l'm)\,x - (mn' - m'n)\,z = dm' - d'm,$$
$$(mn' - m'n)\,y - (nl' - n'l)\,x = dn' - d'n.$$

Squaring and adding, we have, by (1),

$$(x^2 + y^2 + z^2)\sin^2\theta - \{(mn' - m'n)\,x + (nl' - n'l)\,y + (lm' - l'm)\,z\}^2$$
$$= (dl' - d'l)^2 + (dm' - d'm)^2 + (dn' - d'n)^2$$
$$= d^2 - 2dd' \cos\theta + d'^2 = \delta^2,$$

if we suppose (18) and (19) to define two planes, θ to represent the angle between the perpendiculars d and d', and δ the side opposite.

Further reducing by (20), we ultimately find

$$(mn' - m'n)\,x + (nl' - n'l)\,y + (lm' - l'm)\,z = \{(\sin\theta)^2 - \delta^2\}^{\frac{1}{2}} \text{...} (21).$$

Eliminating by cross multiplication between (18), (19), and (21), we have, still paying attention to (1),

$$(\sin\theta)^2 x = d\,\{m'(lm' - l'm) - n'(nl' - n'l)\}$$
$$+ d'\,\{n(nl' - n'l) - m(lm' - l'm)\}$$
$$+ (mn' - m'n)\sqrt{\{(\sin\theta)^2 - \delta^2\}}$$
$$= l(d - d'\cos\theta) + l'(d' - d\cos\theta) + (mn' - m'n)\sqrt{\{(\sin\theta)^2 - \delta^2\}},$$

after effecting some obvious reductions.

Hence

$$\left.\begin{aligned} x&=\frac{l(d-d'\cos\theta)+l'(d'-d\cos\theta)+(mn'-m'n)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \\ y&=\frac{m(d-d'\cos\theta)+m'(d'-d\cos\theta)+(mn'-m'n)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \\ z&=\frac{n(d-d'\cos\theta)+n'(d'-d\cos\theta)+(lm'-l'm)\sqrt{\{(\sin\theta)^2-\delta^2\}}}{(\sin\theta)^2} \end{aligned}\right\} \quad \ldots\ldots(22).$$

6. As an application, suppose it required to determine the path of a given ray of light, after refraction at a proposed surface, the refractive indices of the media being μ and μ_1 respectively.

Let l, m, n, be the direction cosines of the ray before refraction, l_1, m_1, n_1, those after refraction, L, M, N, those of the tangent plane to the surface at the point of incidence, i the angle of incidence, i_1 that of refraction; both of which latter may be considered as known quantities, so that our first equation will be

$$Ll_1 + Mm_1 + Nn_1 = \cos i_1.$$

Moreover, since the incident ray, the refracted ray, and the normal to the surface, lie in the same plane,

$$\frac{(Mn-nM)\,l_1 + (Nl-Ln)\,m_1 + (Lm-lM)\,n_1}{\sin i} = 0,$$

the denominator $\sin i$* being here necessary, in order that the coefficients of the unknown quantities may satisfy the condition, to which we have supposed them subject in the equations of the preceding section.

Hence, by comparison with (22),

$$l_1 = L\cos i_1 \pm \frac{M(Lm-Ml)-N(Nl-Ln)}{\sin i}\sin i_1$$

$$= L\cos i_1 \mp \frac{(l-L\cos i)}{\mu}\,\mu_1,$$

$$\therefore\ (l_1 - L\cos i_1)\,\mu = \mp(l-L\cos i)\,\mu_1,$$

$$\text{or}\quad l_1\mu - l\mu_1 = L(\mu\cos i_1 - \mu_1\cos i),$$

since, as is evident, the lower sign in the preceding equation is to be used.

$$\text{Similarly,}\quad m_1\mu + m\mu_1 = M(\mu\cos i_1 - \mu_1\cos i),$$

$$n_1\mu - n\mu_1 = N(\mu\cos i_1 - \mu_1\cos i),$$

which give the relations sought. It may here be seen, that if $F=0$ be the equation of the surface,

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* $\sin i = \sqrt{\{(Mn-Nm)^2+(Nl-Ln)^2+(Lm-Ml)^2\}}$.

$$\frac{\frac{dF}{dx}}{l\mu_1 - l_1\mu} = \frac{\frac{dF}{dy}}{m\mu_1 - m_1\mu} = \frac{\frac{dF}{dz}}{n\mu_1 - n_1\mu} \ldots\ldots(23),$$

and that in the case of reflection, to which our results may be made to apply by the assumption of $\mu_1 = -\mu$,

$$\frac{\frac{dF}{dx}}{l+l_1} = \frac{\frac{dF}{dy}}{m+m_1} = \frac{\frac{dF}{dz}}{n+n_1} \ldots\ldots\ldots(24).$$

7. The investigation of the curvature of surfaces, considered in its most general aspect, may be greatly simplified by the adoption of processes analogous to those which we have already employed. This we shall proceed to exemplify in a brief discussion of that subject. The course we shall follow, as the most philosophical, will be, to determine the value of the radius of curvature generally, in terms of the differentials of the variables, *i. e.* of the relative indices of position of three consecutive points of the curve, and afterwards to effect those transformations which are necessary to reduce it into an interpretable form. The conditions of the problem mathematically expressed, are

$$(x-x')\,dx + (y-y')\,dy + (z-z')\,dz = 0 \ldots\ldots\ldots(25),$$

$$(x-x')\,d^2x + (y-y')\,d^2y + (z-z')\,d^2z = -ds^2 \ldots(26),$$

$$(x-x')\,X + (y-y')\,Y + (z-z')\,Z = 0 \ldots\ldots\ldots(27),$$

where

$$X = dy\,d^2z - dz\,d^2y, \quad Y = dz\,d^2x - dx\,d^2z, \quad Z = dx\,d^2y - dy\,d^2x;$$

and from these we are to seek the ratio of the function

$$\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}.$$

From (25) and (26) eliminate in succession $x - x'$, $y - y'$, $z - z'$, and we obtain

$$(z-z')\,Y - (y-y')\,Z = dx\,ds^2 \ldots\ldots\ldots\ldots(28),$$

$$(x-x')\,Z - (z-z')\,X = dy\,ds^2 \ldots\ldots\ldots\ldots(29),$$

$$(y-y')\,X - (x-x')\,Y = dz\,ds^2 \ldots\ldots\ldots\ldots(30),$$

$(27)^2 + (28)^2 + (29)^2 + (30)^2$ gives

$$\{(x-x')^2 + (y-y')^2 + (z-z')^2\}\,(X^2 + Y^2 + Z^2) = ds^6,$$

$$\therefore\ R = \frac{ds^3}{\sqrt{(X^2+Y^2+Z^2)}} \ldots\ldots\ldots(31),$$

which is the well known theorem.

8. This expression it only remains to transform into another, which shall involve the partial differential coefficients of two known functions of the variables, instead of the increments of the variables themselves. The process of elimination, by ... this may be

effected, it will be unnecessary here to detail, and I shall therefore simply present the results.

Let the equations of the proposed curve in space, however formed, be represented by

$$F = 0, \quad F' = 0,$$

and for the present, suppose the second of these equations linear, and of the form

$$l(x-a)+m(y-b)+n(z-c)=0;$$

then, after effecting our transformations by the aid of the differential equations of the first and second order, we find, representing the particular value of R thus obtained by r,

$$r=\frac{\left\{\left(l\frac{dF}{dy}-m\frac{dF}{dx}\right)^2+\left(m\frac{dF}{dz}-n\frac{dF}{dy}\right)^2+\left(n\frac{dF}{dx}-l\frac{dF}{dz}\right)^2\right\}^{\frac{3}{2}}}{\left\{\left(n\frac{dF}{dx}-l\frac{dF}{dz}\right)\frac{d}{dy}+\left(m\frac{dF}{dz}-n\frac{dF}{dy}\right)\frac{d}{dx}+\left(l\frac{dF}{dy}-m\frac{dF}{dx}\right)\frac{d}{dz}\right\}^2 F} \quad \ldots\ldots (32),$$

the denominator of which is to be developed and applied, according to the laws of the separation of symbols. This expression may be reduced, by aid of the theorem (1), to the more simple form

$$r=\frac{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}}}{\left(l_1\frac{d}{dx}+m_1\frac{d}{dy}+n_1\frac{d}{dz}\right)^2 F}\sin\theta,$$

wherein θ is the inclination of the tangent plane of F to F′ and l_1, m_1, n_1, the direction cosines of the line of their intersection.

9. When neither F nor F′ is supposed to be linear, the result of the transformation will be more complicated, but may by comparison with (32) be reduced to a simpler form, leading to the following theorem. If r be the radius of curvature of the surface F relative to the tangent plane of the surface F′, and r' that of F′ relative to the tangent plane of F, and θ the inclination of those planes to each other, then putting R for the radius of curvature of the intersection of F and F′ at the same point

$$\frac{1}{R}=\sqrt{\left(\frac{1}{r^2}-\frac{2\cos\theta}{rr'}+\frac{1}{r'^2}\right)}.$$

As $\frac{1}{R}$ is the measure of curvature, it may be replaced by C; and, in like manner, $\frac{1}{r}$ and $\frac{1}{r'}$ by c and c', so that our theorem becomes

$$C=\sqrt{(c^2-2cc'\cos\theta+c'^2)},$$

which, considering the nature of the subject, is a form of great simplicity.

10. The general theorem for minimum distances above given, and those for the solution of the system of equations,

$$ax + by + cz = d,$$
$$a'x + b'y + c'z = d',$$
$$x^2 + y^2 + z^2 = 1,$$

enable us, in most instances, to express at once the solution of problems on the point, straight line, and plane, from a mere inspection of the conditions. Of this we shall give one further illustration, in the discussion of a very important elimentary problem of Mechanics—the determination of the moment of any system of forces in a rigid system, around a given axis. In this case, as in many others, it will be found that the most general method of resolving the problem is also the most simple, and that it leads to analytical results of great elegance.

Let P be the intensity of a given force, acting on the point a, b, c, in the direction l, m, n, to determine its moment around an axis defined by the equation

$$\frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'} \quad \ldots\ldots (1),$$

the equation of the line in which P is directed will evidently be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \ldots\ldots (2),$$

and its effect will not be altered, if we suppose it applied in the same line (2), at the extremity of the minimum distance between (1) and (2), the value of which, as given by our theorem, is

$$D = \frac{(a-a')(mn'-m'n)+(b-b')(nl'-n'l)+(c-c')(lm'-l'm)}{\sqrt{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}}} \quad \ldots (3).$$

Suppose further, the force P resolved into three perpendicular directions, one of them parallel to the axis (1), a second in the line of minimum distance, and the third perpendicular to the two; the cosines of the angles which these directions make with (2) we shall call α, β, γ. Then, as is evident,

$$\alpha = ll' + mm' + nn' = \cos\theta,$$
$$\beta = 0$$

also $$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

whence $\gamma = \sqrt{(1-\alpha^2)} = \sin\theta$, if θ be the angle between (1) and (2). Now the moment sought is evidently $P\gamma \times D$, and since the denominator of D (3) is equal to $\sin\theta$, we obtain, on effecting the substitutions,

$$M = P\{(a-a')(mn'-m'n)+(b-b')(nl'-n'l) + (c-c')(lm'-l'm)\} \quad ..(4),$$

or in terms of D the minimum distance of the given lines, and θ the angle between them,

$$M = PD \sin \theta.$$

If we wish to find the amount of P around an axis passing through the same point a', b', c', and parallel to the axis of x, we must in (4) assume

$$l' = 1, \quad m' = 0, \quad n' = 0,$$

and calling this moment A we have

$$A = P\{(b - b')\,n - (c - c')\,m\}.$$

Similarly the moments B and C around axes passing through the point a, b, c, and parallel to y and z respectively, will be

$$B = P\{(c - c')\,l - (a - a')\,n\}$$
$$C = P\{(a - a')m - (b - b')\,l\};$$

on comparing these expressions with (4) we see at once that

$$M = Al' + Bm' + Cn' \ldots\ldots (5),$$

and from the linear form of the equation it is evident that it will hold true if M, A, B, C, represent the sums of the moments of any number of forces around the same axes. From this expression the value of the maximum moment, and other similar particulars, may be deduced in the ordinary way.

VIII.—ON THE EQUATION OF PAYMENTS.

We propose in the present paper to investigate the problem of the equation of payments, both for simple and for compound interest. We will commence with the case of simple interest; and, in the first place, we will suppose that there are only two sums; which corresponds to the following problem. A owes B two sums of money s_1 and s_2, due respectively at the end of t_1 and t_2 years. He agrees to pay B both sums together at the end of T_2 years, where T_2 is greater then t_1 and less than t_2. Supposing this arrangement to be equitable, determine the value of T_2.

Let $r =$ the rate of interest.

Then, since B receives the sum s_1 too late by $T_2 - t_1$ years, he loses in this respect interest to the amount of $rs_1(T_2 - t_1)$, and since he receives the sum s_2 too early by $t_2 - T_2$ years, he gains in this case the discount of s_2 for the $t_2 - T_2$ years, which is equal to

$$rs_2(t_2 - T_2) : \{1 + r(t_2 - T_2)\}.$$

Hence manifestly, that the arrangement may be equitable, we must have

$$rs_1(T_2 - t_1) = \frac{rs_2(t_2 - T_2)}{1 + r(t_2 - T_2)},$$

and therefore

$$rs_1(T_2 - t_1)(T_2 - t_2) = s_1(T_2 - t_1) + s_2(T_2 - t_2) \ldots\ldots (1).$$

This equation, being a quadratic in T_2, will have two different solutions. Before proceeding further, we will explain the origin of the solution which is foreign to the present question. For this purpose we enunciate the following problem.

A owes B two sums of money s_1 and s_2, due respectively at the end of t_1 and t_2 years. He pays B both sums together after a lapse of T_2 years, where T_2 is greater than t_1 and than t_2; and in order to compensate him for the loss of interest which he has incurred upon the two sums, he consents to pay him, in addition, the interest for $T_2 - t_2$ years of the interest which he has lost upon the sum s_1. Determine the value of T_2 that the bargain may be equitable.

The additional sum which A pays B is clearly

$$rs_1(T_2 - t_1)\, r(T_2 - t_2),$$

and the interest which B has lost, owing to the delay in the payment of the two sums, is

$$rs_1(T_2 - t_1) + rs_2(T_2 - t_2);$$

hence, that the agreement may be equitable, we must have

$$rs_1(T_2 - t_1)\, r(T_2 - t_2) = rs_1(T_2 - t_1) + rs_2(T_2 - t_2),$$

or $$rs_1(T_2 - t_1)(T_2 - t_2) = s_1(T_2 - t_1) + s_2(T_2 - t_2),$$

which is identical with the equation (1).

Thus we see that the algebraical solution of either of the two problems will necessarily give rise to two values of T_2, of which the inapplicable one belongs to the conjugate problem. Moreover, it is evident from the nature of the two problems, that the less value of T_2 belongs to the question with which we are at present concerned; and that consequently, if we suppose the values of T_2 to be $\alpha \pm \sqrt{\beta}$, the negative sign must be taken. The general expression, however, for T_2, is ugly and complicated; and, in order to obtain a convenient approximate formula, it is customary in practice to neglect the term involving r in the equation (1) as small, whence

$$0 = s_1(T_2 - t_1) + s_2(T_2 - t_2),$$

or $$(s_1 + s_2)\, T_2 = s_1 t_1 + s_2 t_2 \ldots\ldots\ldots\ldots (2).$$

Suppose now that T_n denotes the equated time of paying n sums $s_1, s_2, s_3, \ldots s_n$, due respectively at the end of $t_1, t_2, t_3, \ldots t_n$, years, where t_2 is greater than t_1, t_3 than t_2, and so on. Then, since evidently T_n must be the same function of

$$s + s_2 + s_3 + \ldots + s_{n-1},\ s_n,\ T_{n-1},\ t_n,$$

which T_2 is of s_1, s_2, t_1, t_2, respectively, we have by (2), putting $K_n = s_1 + s_2 + s_3 + \ldots + s_n$,

$$K_n T_n = K_{n-1} T_{n-1} + s_n t_n;$$

and therefore, putting for n successively $n - 1$, $n - 2$, $n - 3$, ... we have

$$K_{n-1} T_{n-1} = K_{n-2} T_{n-2} + s_{n-1} t_{n-1}$$
$$K_{n-2} T_{n-2} = K_{n-3} T_{n-3} + s_{n-2} t_{n-2}$$
$$\ldots \ldots\ldots = \ldots\ldots \ldots\ldots$$
$$K_2 T_2 = K_1 T_1 + s_2 t_2;$$

and therefore, by adding together these $n - 1$ equations, and omitting on each side of the result those terms which are common to both, we have, since $K_1 T_1 = s_1 t_1$,

$$K_n T_n = P_n \ldots\ldots (3).$$

where $P_n = s_1 t_1 + s_2 t_2 + s_3 t_3 + \ldots + s_n t_n$.

But if either $T_2 - t_1$, or $t_2 - T_2$ be otherwise than moderately small, it is evident from the equation (1) that the approximation of the equation (2) becomes perfectly inadmissible; and thus we see, that unless the intervals between the epochs of the debts

$$s_1, s_2, s_3, \ldots\ldots s_n,$$

be moderately small, the equation (3) deviates utterly from the truth. In fact, suppose that $t_n = \infty$, while $t_1, t_2, t_3, \ldots t_{n-1}$, are finite: then clearly $P_n = \infty$, and therefore $T_n = \infty$; but it is clear that, since a debt due after an infinite number of years ought to be regarded as no debt at all, we ought to have $T_n = T_{n-1}$.

We will now proceed to the determination of the equated time of payment of any number of debts at compound interest, supposing interest to become principal at every indefinitely small portion of a year. In this case, commencing with only two sums, we have, equating interest lost to discount gained,

$$(\varepsilon^{(T_2 - t_1) r} - 1) s_1 = (1 - \varepsilon^{-(t_2 - T_2) r}) s_2$$
$$= (1 - \varepsilon^{(T_2 - t_2) r}) s_2,$$

therefore $(s_1 + s_2) \varepsilon^{-T_2 r} = \varepsilon^{-t_1 r} s_1 + \varepsilon^{-t_2 r} s_2$;

or putting $\varepsilon^{-r} = \beta$ and $s_1 + s_2 + s_3 + \ldots + s_n = K_n$,

$$K_2 \beta^{T_2} = \beta^{t_1} s_1 + \beta^{t_2} s_2;$$

and therefore, since T_n must evidently be the same function of K_{n-1}, s_n, T_{n-1}, t_n, which T_2 is of s_1, s_2, t_1, t_2, respectively, we have, putting for n successively 3, 4, 5, ... n,

$$K_3 \beta^{T_3} = \beta^{T_2} K_2 + \beta^{t_3} s_3,$$
$$K_4 \beta^{T_4} = \beta^{T_3} K_3 + \beta^{t_4} s_4,$$
$$\ldots\ldots = \ldots\ldots \ldots\ldots$$
$$K_n \beta^{T_n} = \beta^{T_{n-1}} K_{n-1} + \beta^{t_n} s_n:$$

and therefore, by addition and simplification,

$$K_n\beta^{T_n} = \beta^{t_1}s_1 + \beta^{t_2}s_2 + \beta^{t_3}s_3 + \dots + \beta^{t_n}s_n,$$

or putting for β its value

$$\frac{K_n}{\varepsilon^{rT_n}} = \frac{s_1}{\varepsilon^{rt_1}} + \frac{s_2}{\varepsilon^{rt_2}} + \frac{s_3}{\varepsilon^{rt_3}} + \dots + \frac{s_n}{\varepsilon^{rt_n}},$$

which gives us the required expression for T_n.

If the interest be combined with the principal only at the end of each year, we must put $1+r$ in place of ε^r throughout, and thus we have

$$K_n(1+r)^{-T_n} = s_1(1+r)^{-t_1} + s_2(1+r)^{-t_2} + s_3(1+r)^{-t_3} + \dots + s_n(1+r)^{-t_n}.$$

W. W.

IX.—MATHEMATICAL NOTES.

1. *Expression for the Radius of Curvature in Polar Co-ordinates.*

Let ϕ be the angle of contingence, s the arc; then it is known that $\rho = -\dfrac{ds}{d\phi}$, taking the negative sign because, the curve being concave, ϕ diminishes as s increases: and it is required to change this into a function of r and θ.

Now $\dfrac{ds}{d\phi} = \dfrac{ds}{d\theta}\cdot\dfrac{d\theta}{d\phi}$; and as $r\dfrac{d\theta}{dr}$ is the tangent of the angle which the tangent makes with the radius vector,

$$\phi = \pi - \theta - \tan^{-1} r\frac{d\theta}{dr} + \text{const.},$$

the constant depending on the position of the line from which θ is measured. Hence

$$\frac{d\phi}{d\theta} = -\left\{1 + \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\right\}$$

$$= -\frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2},$$

and as $\dfrac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}$,

$$\rho = \frac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

Again,
$$\frac{1}{\rho} = -\frac{d\phi}{ds}$$
$$= -\frac{d}{ds}\left(\tan^{-1}\frac{dy}{dx}\right)$$
$$= \frac{d^2x\,dy - d^2y\,dx}{ds^3}.$$

σ.

2. *The following is a short demonstration of a property of Laplace's Functions.*

Let $$\text{R} = \frac{1}{\sqrt{(r^2 + r'^2 - 2prr')}} = \Sigma\left(\text{Q}_i\,\frac{r'^i}{r^{i+1}}\right).$$

It is clear that when $r = r'$, $\dfrac{d\text{R}}{dr} = \dfrac{d\text{R}}{dr'}$.

Now, when $r = r'$, $$\frac{d\text{R}}{dr} = -\frac{1}{r^2}\,\Sigma.(i+1)\,\text{Q}_i,$$
$$\frac{d\text{R}}{dr'} = \frac{1}{r^2}\,\Sigma.\,i\,\text{Q}_i,$$
$$\text{R} = \frac{1}{r}\,\Sigma\text{Q}_i,$$

therefore $$2\frac{d\text{R}}{dr} + \frac{1}{r}\text{R} = 0.$$

Also from the first two equations it appears that
$$\Sigma.(2i+1)\,\text{Q}_i = 0,$$
the well-known property of these functions.

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I.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS. No. II.

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IN the last number of the Journal, a method was given for the investigation of a class of differential equations, by means of successive reductions.

The present communication contains solutions of some analogous equations effected by a similar process. The results will however be exhibited in a very different form.

We begin by taking a particular case of the equations in question,

$$\frac{d^m y}{dx^m} + ky = \frac{pm}{x}\,\frac{d^{m-1}y}{dx^{m-1}} \;\ldots\ldots\; (1),$$

where p is an integer.

Let $y = \Sigma a_n x^n$,

$$\therefore\; n(n-1)\ldots(n-m+2)(n-m+1-pm)\,a_n + k\,a_{n-m} = 0 \;\ldots\; (2).$$

Assume

$$a_n = \{n-m+1-(p-1)m\}\;\{n-m+1-(p-2)m\}$$
$$\ldots(n-m+1)f(k)\,b_n \ldots\ldots (3),$$

$f(k)$ being some function of k, to be determined hereafter. Then

$$a_{n-m} = (n-m+1-pm)\;\{n-m+1-(p-1)m\}$$
$$\ldots(n-m+1-m)f(k)\,b_{n-m} \ldots (4).$$

If we substitute these values in (2), every factor of (4) and every factor, except the last, of (3), will disappear, and the resulting equation will be

$$n(n-1)\ldots(n-m+2)(n-m+1)\,b_n + k\,b_{n-m} = 0 \ldots (5).$$

This is what (2) would be, were $p=0$. Hence $y = \Sigma . b_n x^n$ fulfils the equation $\frac{d^m y}{dx^m} + ky = 0$, to which (1) would, in that case, be reduced. Let $y = \mathrm{X}$ be the ordinary form of the solution of the last-written equation. Then we must obviously have

$$\Sigma . b_n x^n = \phi(k) . \mathrm{X},$$

where $\phi(k)$ may be any function of k. A very little attention to the mode of integrating linear equations with constant coefficients will show, that in X, x always occurs in conjunction with $k^{\frac{1}{m}}$.

If we put $$\mathrm{X} = \Sigma . \mathrm{A}_n x^n,$$

we must consequently have

$$\mathrm{A}_n = \mathrm{N} k^{\frac{n}{m}},$$

where N is a function of n, except for values of $n < m$, when it is an arbitrary constant; therefore

$$b_n = \mathrm{N}\,\phi(k)\,k^{\frac{n}{m}}.$$

Recurring to (3), inverting the factors and multiplying and dividing by m^p, we shall easily deduce the following equation,

$$a_n = m^p \left(\frac{n-m+1}{m}\right)\left(\frac{n-m+1}{m}-1\right) \cdots \left(\frac{n-m+1}{m}-(p-1)\right) \mathrm{N} . f(k)\,\phi(k)\,k^{\frac{n}{m}}.$$

The form of these p decreasing factors naturally suggests the idea of making $f(k) = k^{-p}$; and if we then put $\phi(k) = k^{-\frac{m-1}{m}}$, we get

$$a_n = m^p \left(\frac{n-m+1}{m}\right)\left(\frac{n-m+1}{m}-1\right) \cdots \left(\frac{n-m+1}{m}-(p-1)\right) \mathrm{N} k^{\frac{n-m+1}{m}-p};$$

$$\therefore\ a_n = m^p \frac{d^p}{dk^p} b_n, \text{ and } \therefore\ \Sigma a_n x^m = m^p \frac{d^p}{dk^p} \Sigma b_n x^n.$$

The factor m^p may obviously be neglected, and we shall therefore have, on replacing $\Sigma b_n x^n$ by $\phi(k)\mathrm{X}$, *i.e.* by $\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}$, the following equation,

$$y = \frac{d^p}{dk^p} \frac{\mathrm{X}}{k^{\frac{m-1}{m}}},$$

for the solution of (1), $y = \mathrm{X}$ being that of $\frac{d^m y}{dx^m} + ky = 0$.

If $m=2$, $\mathrm{X}=\mathrm{C}\sin\{\sqrt{(k)}\,x+a\}$, and $\dfrac{m-1}{m}=\frac{1}{2}$. Hence

$$y=\frac{d^p}{dk^p}\,\frac{\mathrm{C}\sin\{\sqrt{(k)}\,x+a\}}{\sqrt{(k)}}$$

is the solution of

$$\frac{d^2y}{dx^2}+ky=\frac{2p}{x}\,\frac{dy}{dx}.$$

This result is given in *Hymers'* Diff. Equations, and is, I believe, due to Mr. Gaskin.

If the proposed equation were

$$\frac{d^my}{dx^m}+ky+\frac{pm}{x}\,\frac{d^{m-1}y}{dx^{m-1}}=0,$$

we should immediately conclude, from analogy, that its solution must be

$$y=\frac{d^{-p}}{dk^{-p}}\,\frac{\mathrm{X}}{k^{\frac{m-1}{m}}};$$

but it may be as well to establish this conclusion by an independent investigation.

Equation (2) will, in this case, be

$$n(n-1)\ldots(n-m+2)(n-m+1+pm)\,a_n+k\,a_{n-m}=0.$$

Assume

$$a_n=f(k)\,\frac{b_n}{(n-m+1+pm)\{n-m+1+(p-1)m\}\ldots(n-m+1+m)},$$

$$\therefore\ a_{n-m}=f(k)\,\frac{b_{n-m}}{\{n-m+1+(p-1)\,m\}\,\ldots(n-m+1)},$$

therefore $n(n-1)\ldots(n-m+1)\,b_n+kb_{n-m}=0.$

Therefore as before,

$$b_n=\mathrm{N}\,.\,\phi(k)\,k^{\frac{n}{m}},$$

$$\text{and } a_n=m^{-p}\,\mathrm{N}\,\frac{f(k)\,\phi(k)\,k^{\frac{n}{m}}}{\left(\dfrac{n-m+1}{m}+1\right)\cdots\left(\dfrac{n-m+1}{m}+p\right)}.$$

Here we make $f(k)=k^p$, and $\phi(k)=k^{-\frac{m-1}{m}}$,

$$\therefore\ a_n=m^{-p}\,\mathrm{N}\,\frac{d^{-p}}{dk^{-p}}\,b_n,$$

no complementary function being added.

Hence, precisely as before, we find that

$$y=\frac{d^{-p}}{dk^{-p}}\,\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}$$

is the solution required.

Let us now consider the more general equation,

$$\frac{d^m y}{dx^m} + ky = pm \frac{d^{m-s-1}}{dx^{m-s-1}} \left(\frac{1}{x} \frac{d^s y}{dx^s}\right) \ldots\ldots (6).$$

If $y = \Sigma \, . \, a_n x^n$, there will be

$$n \ldots (n-s+1)(n-s-pm)(n-s-1)\ldots$$
$$\ldots (n-m+1)\, a_n + k a_{n-m} = 0 \ldots\ldots (7).$$

This equation is analogous to (2); but $m-1$ is replaced by s. Assume, therefore,

$$a_n = \{n-s-(p-1)m\}\{n-s-(p-2)m\} \ldots (n-s)\, k^{-p} . b_n \ldots (8),$$
$$\therefore\ a_{n-m} = (n-s-pm)\{n-s-(p-1)m\} \ldots\ldots (n-s+m)\, k^{-p} b_{n-m},$$

and $n \ldots (n-s+1)(n-s)(n-s-1)\ldots(n-m+1) b_n + k\, b_{n-m} = 0.$

Hence, as before,

$$\Sigma b_n x^n = \phi(k)\, \mathrm{X},$$

where X denotes the same function of x that it did in the former case.

Consequently $\qquad b_n = \mathrm{N}\, \phi(k)\, k^{\frac{n}{m}}$, and

$$a_n = m^p \left(\frac{n-s}{m}\right)\left(\frac{n-s}{m} - 1\right) \ldots \left\{\frac{n-s}{m} - (p-1)\right\} \mathrm{N}\, \phi(k)\, k^{\frac{n}{m} - p}.$$

We must, it is evident, make $\phi(k) = k^{-\frac{s}{m}}$, and then

$$a_n = m^p \frac{d^p}{dk^p} \mathrm{N}\, k^{\frac{n-s}{m}} = m^p \frac{d^p}{dk^p} b_n .$$

Hence $y = \dfrac{d^p}{dk^p} \dfrac{\mathrm{X}}{k^{\frac{s}{m}}}$ is the solution of the equation

$$\frac{d^m y}{dx^m} + ky = pm \frac{d^{m-s-1}}{dx^{m-s-1}} \left(\frac{1}{x} \frac{d^s y}{dx^s}\right),$$

for as before the factor m^p may be neglected.

A particular case of this result is that in which $s = 0$. If with this value of s we have $m = 2$, the equation to be integrated takes the form

$$\left(\frac{d^2 y}{dx^2} + ky\right) x^2 = 2p \left(x \frac{dy}{dx} - y\right),$$

and the solution is

$$y = \mathrm{C} \frac{d^p}{dk^p} \sin \{\surd(k)\, x + \alpha\}.$$

Equation (7) is the most general one in which the coefficient of a_n differs in one factor only from what it is in the case of

$$\frac{d^m y}{dx^m} + ky = 0.$$

But our method is applicable in other cases.

Let us resume the equation discussed in the last number of the Journal,

$$\frac{d^m y}{dx^m} + ky = p(p-1)\frac{1}{x^2}\frac{d^{m-2}}{dx^{m-2}}y \;\ldots\ldots (8).$$

By the usual method of making $y = \Sigma a_n x^n$, we get

$$n(n-1)\,..(n-m+3)\{(n-m+2)(n-m+1)-p\,.(p-1)\}\,a_n + k\,a_{n-m} = 0,$$

or
$$n(n-1)\ldots(n-m+3)\{(n-m+2-p)(n-m+1+p)\}\,a_n + k\,a_{n-m} = 0 \ldots\ldots (9).$$

It may be remembered that we found it necessary that p or $p-1$ should be divisible by m. Suppose then that p is so divisible, and that the quotient is q.

Let

$$a_n = fk\,\frac{\{(n-m+2-(p-m)\}\,\{n-m+2-(p-2m)\}\ldots(n-m+2)}{(n-m+1+p)\{n-m+1+(p-m)\}\ldots(n-m+1+m)}\,b_n \;\ldots\ldots (10),$$

$$\therefore\; a_{n-m} = fk\,\frac{(n-m+2-p)\,\{n-m+2-(p-m)\}\ldots\ldots\ldots}{\{n-m+1+(p-m)\}\ldots\ldots\ldots(n-m+1)}\,b_{n-m},$$

(there are q factors in both numerator and denominator); equation (9) becomes

$$n(n-1)\,..(n-m+3)(n-m+2)(n-m+1)\,b_n + k\,b_{n-m} = 0;$$

and, as in the two preceding cases, we shall have

$$\Sigma b_n x^n = \phi(k)\,\mathrm{X},$$

$$\text{and}\qquad b_n = \mathrm{N}\,.\,k^{\frac{n}{m}}\phi(k).$$

(10) may be written thus, as $p = qm$,

$$a_n = f(k)\,\frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,b_n.$$

Now
$$\frac{d^q}{dk^q}\,k^{\frac{n-m+2}{m}} = \left(\frac{n-m+2}{m}\right)\ldots\left(\frac{n-m+2}{m}-q+1\right)k^{\frac{n-m+2}{m}-q};$$

therefore
$$\frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,.\,k^{\frac{n-m+2}{m}}\right)$$

$$= \frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,.\,k^{\frac{n-m+1}{m}+q}.$$

But if we make $\phi(k) = k^{\frac{-m+2}{m}}$, then

$$b_n = \mathrm{N} \,.\, k^{\frac{n-m+2}{m}}.$$

Let us also put $f(k) = k^{q-\frac{1}{m}}$; therefore

$$a_n = \frac{\left(\frac{n-m+2}{m}\right)\ldots\&c.}{\left(\frac{n-m+1}{m}+1\right)\ldots\&c.}\,\mathrm{N}\,k^{\frac{n-m+1}{m}+q}$$

$$= \mathrm{N}\,\frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,k^{\frac{n-m+2}{m}}\right),$$

$$\text{or}\quad a_n = \frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,b_n\right).$$

Hence
$$y = \frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,\frac{\mathrm{X}}{k^{\frac{m-2}{m}}}\right)\;\ldots\ldots\;\ldots\ldots\;(11),$$

is the solution required.

It admits also of another form, which it may be worth while to remark.

$$\frac{d^{-q}}{dk^{-q}}\,k^{\frac{n-m+1}{m}} = \frac{1}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,k^{\frac{n-m+1}{m}+q},$$

$$\text{therefore}\;\frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,k^{\frac{n-m+1}{m}}\right)$$

$$= \frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,k^{\frac{n-m+2}{m}-q}.$$

Here we must make $\phi(k) = k^{\frac{-m+1}{m}}$ and $f(k) = k^{-q+\frac{1}{m}}$, and then

$$b_n = \mathrm{N}\,k^{\frac{n-m+1}{m}}\quad\text{and}\quad a_n = \mathrm{N}\,(\ldots\ldots)\,k^{\frac{n-m+2}{m}-q},$$

$$\text{therefore}\;a_n = \frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,b_n\right)$$

$$\text{and}\;y = \frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}\right)\;\ldots\ldots\ldots\;(12).$$

The value of y, deduced from the developement of (12), can of course differ only in a factor of some function of k from that

which is given by (11), and it will easily appear, on comparing the values of a_n in the two cases, that this factor is $k^{-2q+\frac{1}{m}}$.

Let us now consider the case in which p is not divisible by m, while $p-1$ is so. And let $p-1=qm$. The two factors of (9) on which our reduction operates, viz.

$$(n-m+2-p)\ (n-m+1+p,)$$

may be written thus,

$$\{n-m+2+(p-1)\}\ \{n-m+1-(p-1)\},$$

$$\text{or}\quad (n-m+2+qm)\ (n-m+1-qm).$$

The change which this will introduce in the process, is not difficult to perceive. We must assume

$$a_n = f(k)\frac{\frac{n-m+1}{m}\ \ldots\ldots\ \frac{n-m+1}{m}-q+1}{\frac{n-m+2}{m}+1\ \ldots\ldots\ \frac{n-m+2}{m}+q}\,b_n$$

as before, the transformed equation will be

$$n\ldots\ldots(n-m+2)\ (n-m+1)\ b_n+k\,b_{n-m}=0.$$

Now

$$\frac{d^q}{dk^q}k^{\frac{n-m+1}{m}} = \left(\frac{n-m+1}{m}\right)\ldots\ldots\left(\frac{n-m+1}{m}-q+1\right)k^{\frac{n-m+1}{m}-q}$$

$$\text{therefore}\ \frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\frac{d^q}{dk^q}k^{\frac{n-m+1}{m}}\right)$$

$$=\frac{\left(\frac{n-m+1}{m}\right)\ldots\ldots\left(\frac{n-m+1}{m}-q+1\right)}{\left(\frac{n-m+2}{m}+1\right)\ldots\ldots\left(\frac{n-m+2}{m}+q\right)}k^{\frac{n-m+2}{m}+q}.$$

If then we make $f(k)=k^{q+\frac{1}{m}}$ and $\phi(k)=k^{\frac{-m+1}{m}}$, we get

$$b_n = \mathrm{N}\,k^{\frac{n-m+1}{m}};\quad a_n=\mathrm{N}\frac{\left(\frac{n-m+1}{m}\right)\ldots\&\text{c.}}{\left(\frac{n-m+2}{m}+1\right)\ldots\&\text{c.}}k^{\frac{n-m+2}{m}+q},$$

$$\text{or}\ a_n = \mathrm{N}\,\frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\frac{d^q}{dk^q}k^{\frac{n-m+1}{m}}\right).$$

Hence, finally,

$$y = \frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\frac{d^q}{dk^q}\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}\right)\ \ldots\ldots\ldots\ldots\ (13).$$

As an illustration, let us take the case of the equation which occurs in the theory of the figure of the earth,

$$\frac{d^2y}{dx^2} + ky = \frac{6y}{x^2}.$$

Here $m = 2$, $p = 3$, $p - 1 = 2$: hence $q = 1$, and as p is not a multiple of m, the formula (13) is to be used.

It is in this case, as $X = C \sin \{\sqrt{(k)}\, x + a\}$,

$$y = \frac{d^{-1}}{dk^{-1}} \left\{ k^{\frac{3}{2}} \frac{d}{dk} \frac{C \sin \{\sqrt{(k)}\, x + a\}}{k^{\frac{1}{2}}} \right\},$$

$$\frac{d}{dk} \frac{\sin \{\sqrt{(k)}\, x + a\}}{k^{\frac{1}{2}}} = \tfrac{1}{2} \frac{x}{k} \cos \{\sqrt{(k)}\, x + a\} - \tfrac{1}{2} \frac{\sin \{\sqrt{(k)} x + a\}}{k^{\frac{3}{2}}},$$

therefore

$$y = \tfrac{1}{2} C \frac{d^{-1}}{d\, k^{-1}} [x k^{\frac{1}{2}} \cos \{\sqrt{(k)}\, x + a\} - \sin \{\sqrt{(k)}\, x + a\}],$$

or integrating the first term by parts,

$$y = k C \sin \{\sqrt{(k)}\, x + a\} - \tfrac{3}{2} C \frac{d^{-1}}{dk^{-1}} \sin \{\sqrt{(k)}\, x + a\}.$$

But

$$\frac{d^{-1}}{dk^{-1}} \sin \{\sqrt{(k)}\, x + a\} = \frac{2k^{\frac{1}{2}}}{x} \cos \{\sqrt{(k)}\, x + a\} + \frac{2}{x^2} \sin \{\sqrt{(k)}\, x + a\},$$

therefore, if $kC = C_1$,

$$y = C_1 \left[\sin \{\sqrt{(k)}\, x + a\} + \frac{3}{x k^{\frac{1}{2}}} \cos \{\sqrt{(k)}\, x + a\} - \frac{3}{kx^2} \sin \{\sqrt{(k)}\, x + a\}\right],$$

which is the required solution.

Equation (13) corresponds to (11): but there is another form of the solution in the case of $p - 1 = qm$, which we shall just mention, and which is the counterpart of (12). It is

$$y = \frac{d^q}{dk^q} \left(k^{-q-\frac{1}{m}} \frac{d^{-q}}{dk^{-q}} \frac{X}{k^{\frac{m-2}{m}}} \right) \text{.........} \quad (14).$$

It would be a needless repetition to go through the steps which lead to this result.

All the operations indicated in these symbolical solutions are practicable. This will appear by considering the nature of the function X, which, in its most general form, consists of the sum of terms, of which the type is

$$C\, e^{[a+\beta\sqrt{(-1)}]\, k^{\frac{1}{m}} x}.$$

If we make $k = \kappa^m$, this will become

$$C\, e^{[\alpha+\beta\sqrt{(-1)}]\,\kappa x},$$

which may be integrated any number of times for κ, and consequently, if it is multiplied by any rational and integral function of κ, it may still be integrated by parts as often as we please. Now $\dfrac{dk}{k^{\frac{s}{m}}}$ will become $m\kappa^{m-s-1}\,d\kappa$; and as m and s are integral, the method of parts applies, provided s is not greater than $m - 1$, which it is in none of our formula.

Fourier's expression, by means of definite integrals for the i^{th} differential coefficient of any function, would enable us to extend our solutions to the cases in which p is fractional. But merely analytical transformations of the results at which we have arrived are not of much interest, and the methods of effecting them are direct and obvious.

Equation (1) admits of another symbolical solution besides the one already given.

It is easily seen, that if

$$\Sigma\, a_n x^n = x^{mp}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\cdot\frac{d}{dx}\,\frac{1}{x^{m-1}}\ldots\Sigma\, b_n x^n, \ (p \text{ factors}),$$

$$a_n = (n-pm+1)\,\ldots\ldots\,(n-m+1)\,b_n,$$

which is what (3) is, when $f(k) = 1$. ($\dfrac{d}{dx}$ applies to all that follows it.)

Hence we shall clearly have

$$y = x^{pm}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\ldots\ldots\ldots\frac{d}{dx}\,\frac{1}{x^{m-1}}\,X,$$

for the solution of (1).

Similarly, the solution of (6) is

$$y = x^{(p-1)m+(s-1)}.\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\ldots\ldots\,\frac{d}{dx}\,\frac{1}{x^{s}}\,X.$$

Many applications and modifications of the method we have employed, will readily present themselves, but the subject is not of sufficient importance to deserve a fuller discussion. It is not difficult to multiply artifices, by means of which particular equations may be solved, but the results will, generally speaking, be of little value.

II.—ON A SIMPLE PROPERTY OF THE CONIC SECTIONS.*

The property in question is not new, though perhaps the proof here given may be so. I am principally induced to send it to the Journal as an illustration of the following remark.

The fundamental propositions of Geometry have each of them a train of consequences, any one of which might frequently be deduced from other fundamental propositions, but not so easily as from that, which we may therefore call its own. If then a very simple proposition will admit only of very complicated proof, it is open to trial whether the proper fundamental proposition has not been omitted from the elements.

The property of the conic sections above-mentioned is the following:—If the tangents at P and Q meet in the point T, and if S be one of the foci, PT and QT subtend equal angles at S, except only when P and Q are on different branches of an hyperbola; in which case the angles are supplemental. The simplicity of this proposition, compared with the difficulty of its proof by the ordinary properties of the curves, suggested the preceding remark, and on trial it appears that this property of the tangents of a conic section is the immediate consequence of the expression of the criterion that a circle touches four straight lines, which is omitted by Euclid.

Let there be four straight lines, no two of which are parallel. These must form such a figure as fig. 1, which may be considered as containing three four-sided figures, of which the sides are

(AB, BC, CD, DA), (AE, EC, CF, FA),
(BE, ED, DF, FB).

These may be called the convex, the single concave, and the double concave figures.

It is easily shown from the elements, that taking an angle of a triangle A, the circle which touches EA, AD, DE, is a function only of A, and of the excess of EA and AD above DE; while the circle which touches GE, ED, DH, is a function only of A, and of the sum of EA, AD, and DE. Hence the following

THEOREM. A circle touches four straight lines if the sum of alternate sides be the same in the convex or the single concave figure, or if the sum of adjacent sides be the same in the double concave figure.

* From a Correspondent.

Remark. Of the three equations

$$AB + CD = AD + BC, \quad AE + CF = AF + EC,$$
$$EB + BF = ED + DF,$$

any one being true, the others are necessarily true; and, any one of these being true, the circle which touches the four straight lines is inscribed in the convex figure; but the equation

$$BE + ED = BF + FD$$

shows that the circle is in the figure GECFH.

Let E and F be the foci of an ellipse, in which the points B and D lie; a circle can then be inscribed in ABCD, that is, the bisectors of the angles ABC and ADC meet in the same point as the bisectors of AED and AFB. But the first bisectors are the tangents at B and D; whence the property above enunciated follow:

Let E and F be the foci of an hyperbola, and A and C two points on the same branch. Then

$$AE + CF = AF + EC,$$

whence a circle can be inscribed in ABCD, the bisectors of A and C meet in the same point as those of E and F, and the same proposition is established.

Let E and F be the foci of an hyperbola, and B and D points on different branches. We have then

$$BE + ED = BF + FD;$$

a circle can be inscribed in GECFH, and the bisectors of EBF and EDF meet in the same point as those of GED and BFH, from whence it is easily shown that the angles subtended by the tangents at either focus are supplemental.

The case in which two lines of the fundamental figure become parallel, and its application to the parabola, offer no difficulty. One focus being then at an infinite distance, the property can only be obviously true of the other. But if through P, T, and Q, we draw lines parallel to the axis, and consider these lines as making infinitely small angles at the other focus, proportional to their distances from each other, we have the following theorem: If P and Q be two points of a parabola, the tangents to which meet in T, the projections of PT and QT upon any line perpendicular to the axis are equal to one another. This theorem can readily be proved by a more (geometrically) justifiable process.

A. D. M.

III.—ON THE MOTION OF A PENDULUM WHEN ITS POINT OF SUSPENSION IS DISTURBED.

In a former article* we investigated the nature of the mutual action of two pendulums united by any elastic or moveable connexion; we shall here consider more particularly the effect produced on the motion of a simple pendulum by a disturbance of its point of support. As before, we shall suppose the motions to be infinitesimal, in order that the equations may be at all manageable, and also for the sake of simplicity we shall assume the disturbances of the point of suspension to be rectilinear.

I. Let the point of suspension have a horizontal motion parallel to that of the pendulum.

The pendulum being a simple one, let u be the horizontal co-ordinate of the point of suspension, $u + x$ of the ball; let l be the length of the pendulum, and $n^2 = \frac{g}{l}$. Then the equation for the motion of the pendulum is

$$\frac{d^2(u+x)}{dt^2} + n^2 x = 0 \;\ldots\ldots\;..\;(1).$$

Two suppositions may be made regarding the nature of the disturbances of the point of suspension, that is, regarding the motion of u: either it has a vibratory motion independent of the motion of the pendulum or depending on it.

In the first case, when the motion of u is independent of that of the pendulum, u is given simply in terms of t, or

$$u = c\cos(at+\beta)\ldots\ldots(2);$$

whence equation (1) becomes

$$\frac{d^2x}{dt^2} + n^2x - ca^2\cos(at+\beta) = 0\;\ldots\ldots\ldots(3).$$

The solution of this is

$$x = \mathrm{A}\cos(nt+\mathrm{B}) - \frac{ca^2}{a^2-n^2}\cos(at+\beta)\;\ldots\;(4).$$

The motion, therefore, consists of two simple oscillations, which may be considered separately. The one expressed by the term $\mathrm{A}\cos(nt+\mathrm{B})$, is independent of the motion of the point of suspension, and depends only on the length of the pendulum and the original circumstances of the motion; the other, expressed by the term $\frac{ca^2}{a^2-n^2}\cos(at+\beta)$, depends on the motion

* Vol. II. No. IX. p. 120.

of the point of suspension, and is synchronous with it, but both in extent and period is quite independent of the initial circumstances. If we neglect the first term, or the regular motion of the pendulum, we have for the disturbed motion

$$x = -\frac{ca^2}{a^2 - n^2}\cos(at + \beta),$$

$$\text{and}\quad u + x = -\frac{cn^2}{a^2 - n^2}\cos(at + \beta).$$

The motion of a point at a distance s from the ball will be

$$= \frac{c}{n^2 - a^2}\left(n^2 - a^2\frac{s}{l}\right)\cos(at + \beta)$$

That point, therefore, will remain at rest, so far as the disturbed motion is concerned, when

$$a^2 = \frac{l}{s}n^2 = \frac{g}{s},$$

or the distance from the ball of the point at rest, is equal to the length of a simple pendulum vibrating in the same time as the point of support, which might have been anticipated.

The preceding formulæ fail when $a = u$, or when the period of the disturbance is equal to that of the pendulum. In this case the integral of the equation (3) would be of the form

$$x = \mathrm{A}\cos(nt + \beta) - \frac{ct}{2n}\sin(nt + \beta),$$

into which the time enters as a multiplier, so that x increases indefinitely with the time, and the motions are therefore no longer infinitesimal, as was at first supposed, and the original equations are therefore no longer applicable. This change of form in the integral is the analytical indication of a very important fact, viz. that a force, however small, may produce a motion of any extent in any body capable of oscillating, provided that the application of the force be made at intervals, the length of which is equal to the period in which the body would oscillate under the action of gravity. Thus it is, that a stone in a sling may be made to revolve completely round with a great angular velocity, merely by the synchronous motion of the hand; and many similar examples of this fact are constantly presented to our observation. We remember to have seen in a steam-vessel a lamp, which was so hung that its time of oscillation very nearly coincided with the stroke of the engine; the consequence of which was, that though the water was quite smooth, the lamp being set in motion by the reiterated strokes of the piston swung in a large arc, as if the vessel were rolling in a heavy sea.

In the second case, when the motion of the point of suspension depends partly on its own elasticity, and partly on the motion of the ball, we shall have for the equation of its motion

$$\frac{d^2u}{dt^2} + k^2u - ax = 0 \;\ldots\ldots$$

which, combined with the equation

$$\frac{d^2u}{dt^2} + \frac{d^2x}{dt^2} + n^2x = 0 \ldots\ldots\ldots\ldots (1),$$

will determine u and x. To do this, muitiply (5) by $\frac{d^2}{dt^2}$ and (1) by $\left(\frac{d^2}{dt^2} + k^2\right)$, and subtract; then

$$\left\{\left(\frac{d^2}{dt^2} + k^2\right)\left(\frac{d^2}{dt^2} + n^2\right) + a\frac{d^2}{dt^2}\right\}x = 0,$$

$$\text{or} \quad \left(\frac{d^2}{dt^2} + \rho_1^2\right)\left(\frac{d^2}{dt^2} + \rho_2^2\right)x = 0 \ldots\ldots (6),$$

ρ^2_1, ρ^2_2 being the roots of the quadratic equation

$$(\rho^2 - k^2)(\rho^2 - n^2) - a\rho^2 = 0 \ldots\ldots\ldots (7).$$

Hence, integrating (6), we have for the value of x

$$x = \text{A} \cos(\rho_1 t + \alpha) + \text{B} \cos(\rho_2 t + \beta) \ldots (8).$$

To deduce the value of u, subtract (1) from (5); then

$$k^2u = \frac{d^2x}{dt^2} + (n^2 + a)x,$$

and therefore

$$u = \frac{\text{A}}{k^2}(n^2 + a - \rho_1^2)\cos(\rho_1 t + \alpha) + \frac{\text{B}}{k^2}(n^2 + a - \rho_2^2)\cos(\rho_2 t + \beta) \ldots (9).$$

In general k is much greater than n, and a is very small. The equation for determining the two values of ρ^2 may then bè put under the forms

$$\rho_1^2 = n^2 - \frac{a\rho_1^2}{k^2 - \rho_1^2},$$

$$\rho_2^2 = k^2 + \frac{a\rho_2^2}{\rho_2^2 - n^2}.$$

One therefore of the two values of ρ will be a little less than n, and the other a little greater than k. Hence the signs of the coefficients of the first terms in x and u will be the same, and those of the second terms different.

The vibrations of the ball, and of the point of suspension, will thus consist of two parts, which may either co-exist or exist separately. The one part, the argument of which is cos $(\rho_1 t + \alpha)$, is a synchronous vibration of the ball, and the point of suspension, a little slower than the independent motion of the ball, and such that the ball and the point of suspension are always on the same side of the perpendicular, passing through the original position. The other part, the argument of which is cos $(\rho_2 t + \alpha)$, is a synchronous vibration, a little quicker than the independent vibration of the point of support, and such that the ball and the point of suspension are always on opposite sides of the perpendicular.

If instead of a simple pendulum we have a rod or other solid body, let a be the distance of its centre of gravity from the point of suspension, and k the radius of gyration; then if $u+x$ be the co-ordinate of the centre of gravity, and if we suppose the motion of the point of suspension to be independent of that of the pendulum, so that $n = c\cos(\alpha t+\beta)$, we shall have, for determining x, the equation

$$\frac{d^2x}{dt^2}+\frac{ga}{a^2+k^2}x-\frac{a^2}{a^2+k^2}c\alpha^2\cos(\alpha t+\beta)=0 \ldots\ldots (10).$$

Let $\dfrac{ga}{a^2+k^2}=n^2$. Then integrating

$$x=\mathrm{A}\cos(nt+\mathrm{B})-\frac{n^2a^2c\alpha}{g(\alpha^2-n^2)}\cos(\alpha t+\beta)\ldots\ldots(11).$$

Neglecting the first term, we get, as the expression for the part of the motion independent of the initial circumstances,

$$x=-\frac{n^2a^2c\alpha}{g(\alpha^2-n^2)}\cos(\alpha t+\beta).$$

For a point at a distance s from the point of suspension, the co-ordinate is $u+\dfrac{sx}{a}$

$$=\left(1-\frac{n^2}{g}\,\frac{\alpha^2 s}{\alpha^2-n^2}\right)c\cos(\alpha t+\beta).$$

From this it appears that a point, the distance of which from the point of suspension is $\left(a+\dfrac{k^2}{a}\right)\left(1-\dfrac{n^2}{\alpha^2}\right)$ will remain at rest. If $\dfrac{n}{\alpha}=0$, that is, if α be very great, or the vibrations of the point of suspension become very rapid, the centre of percussion is the point which will remain at rest, as might have been anticipated.

II. Let the point of suspension have a horizontal motion at right angles to that of the pendulum.

Without going into the calculation in this case, it is easy to see that this disturbance will not affect the motion of the pendulum in its original direction, and that it will give rise to an oscillatory motion at right angles to that of the pendulum, the nature of which will be similar to the disturbance described in the first case. The two independent motions at right angles to each other will be combined into one, which will cause the ball of the pendulum to describe a curvilinear path.

III. Let the point of suspension have a small vertical oscillatory motion while the ball of the pendulum oscillates in one plane.

In this case, the disturbance of the motion of the pendulum is produced by the variation in the tension of the string; therefore, if

x represent the horizontal co-ordinate of the ball, and T the tension of the string, the equation of motion is

$$\frac{d^2x}{dt^2} + \mathrm{T}\,\frac{x}{l} = 0 \ \ldots\ldots\ (12).$$

Let the vertical motion of the point of suspension be

$$u = c \cos(at + \beta),$$

being independent of the motion of the pendulum. The ball receives the same motion from the change of tension, and therefore

$$\frac{d^2u}{dt^2} = -\,ca^2 \cos(at + \beta) = \mathrm{T} - g;$$

and therefore, putting n^2 for $\frac{g}{l}$, equation (12) becomes

$$\frac{d^2x}{dt^2} + n^2x - c\,\frac{a^2}{l}\,x \cos(at+\beta) = 0 \ \ldots\ldots\ (13).$$

This equation being no longer linear, cannot be integrated as the preceding equations were, and we must therefore have recourse to an approximate solution. If we suppose c to be small, we may substitute in that term the value of x derived from the supposition of $c = 0$. That gives us

$$x = \mathrm{A} \cos(nt + \mathrm{B}),$$

and therefore

$$\frac{d^2x}{dt^2} + n^2x = \frac{\mathrm{A}ca^2}{2l}\left[\cos\{(n+a)\,t+\mathrm{B}+\beta\} + \cos\{(n-a)\,t+\mathrm{B}-\beta\}\right].$$

Integrating this equation,

$$x = -\,\frac{\mathrm{A}ca^2}{2l}\left\{\frac{\cos[(n+a)t+\mathrm{B}+\beta]}{a^2+2an} + \frac{\cos[(n-a)\,t+\mathrm{B}-\beta]}{a^2-2an}\right\} \ldots (14).$$

The most important conclusion from this result is, that if $\mathrm{A} = 0$, or the ball be originally at rest, it will have no motion communicated to it by the vertical motion of the point of suspension: and that if it has an oscillatory motion in any direction, this will not be permanently altered unless $a = 2n$, or the period of oscillation of the pendulum be double of that of the point of suspension. In this case the integral becomes infinite; and if, as before, we put it into another shape, we find that x increases continually with the time. This accords with experiment, which shows that the arc of vibration of a pendulum may be increased indefinitely by giving the point of suspension a vertical motion of oscillation, the period of which is half of that of the pendulum.

G. S.

IV.—ON THE INTEGRATION OF EQUATIONS OF PARTIAL DIFFERENTIALS.

By B. Bronwin.

It is only of linear equations of the second and higher orders that I propose to treat, and that for the sake of noticing the unnatural and preposterous nature of the received Theory with reference to them. For example, let

$$\frac{d^2z}{dx^2} + A\frac{d^2z}{dx\,dy} + B\frac{d^2z}{dy^2} = C;\ \frac{dz}{dx} = p,\ \frac{dz}{dy} = q.$$

From hence is derived

$$dp\,dy + Bdq\,dx - Cdx\,dy = \frac{d^2z}{dx\,dy}(dy^2 - Adx\,dy + Bdx^2).$$

And similarly when there are more variables, or the equation is of a higher order. Now to introduce the differentials of the second and higher degrees into quantities which are to be complete differentials, I call unnatural and preposterous. Next we make $dy = mdx$ to render the equation linear. But this is establishing a relation between two quantities absolutely independent, which is another absurdity. It is very true that in equations of the first order we do make such an assumption; but it is because we find a certain consequent to result from it. $dy - mdx = 0$ leads to $dM = 0$; and we want to find $M = a$. But this appears to me a very different thing from the way in which the assumption is made in the above equation.

The natural mode of treating the subject I conceive to be the following:

$$dp = \frac{d^2z}{dx^2}dx + \frac{d^2z}{dx\,dy}dy,\quad dq = \frac{d^2z}{dx\,dy}dx + \frac{d^2z}{dy^2}dy.$$

Multiply the last of these by the indeterminate m, add the product to the first, and to the sum the given equation multiplied by dx; we obtain

$$dp + mdq - Cdx = \frac{d^2z}{dx\,dy}\{dy + (m - A)\,dx\} + \frac{d^2z}{dy^2}(mdy - Bdx).$$

Now make

$$A - m = \frac{B}{m},\ \text{or}\ m^2 - Am + B = 0;$$

and let n and n' be the roots of this last; also

$$Q = \frac{d^2z}{dx\,dy} + n\frac{d^2z}{dy^2},\quad Q' = \frac{d^2z}{dx\,dy} + n'\frac{d^2z}{dy^2},$$

and the above equation becomes

$$dp + ndq - Cdx = Q\,(dy - ndx);$$

$$\text{or}\quad dp + n'dq - Cdx = Q'\,(dy - n'dx).$$

Here we have only one indeterminate quantity, as in the received Theory, and the differentials are only of the first degree. If the first member be a complete differential, the second must be one also; and we may integrate as in an equation of the first order.

If the first member be not a complete differential, make

$$dM = \omega dN, \text{ or } M = \phi(N),$$

ω being an indeterminate, and M, N functions of x, y, z, p, and q. As x and y are independent, we must have

$$\left(\frac{dM}{dx}\right) = \omega\left(\frac{dN}{dx}\right), \quad \left(\frac{dM}{dy}\right) = \omega\left(\frac{dN}{dy}\right).$$

But, on account of the indeterminate ω, we must have

$$\left(\frac{dM}{dx}\right) = 0, \&c.;$$

$$\text{or } \frac{dM}{dp}\frac{dp}{dx} + \frac{dM}{dq}\frac{dq}{dx} + \frac{dM}{dz}p + \frac{dM}{dx} = 0;$$

$$\frac{dM}{dp}\frac{dp}{dy} + \frac{dM}{dq}\frac{dq}{dy} + \frac{dM}{dz}q + \frac{dM}{dy} = 0.$$

From the first of these we eliminate $\frac{dp}{dx}$ by the given equation, and there results

$$C\frac{dM}{dp} + \left(\frac{dM}{dq} - A\frac{dM}{dp}\right)\frac{dq}{dx} - B\frac{dM}{dp}\frac{dq}{dy} + \frac{dM}{dz}p + \frac{dM}{dx} = 0.$$

If now we multiply the second by the indeterminate m, and add the product to the equation last found; making $A - m = \frac{B}{m}$, or $m^2 - Am + B = 0$, of which the roots are n and n'; we obtain

$$\left(\frac{dq}{dx} + n'\frac{dq}{dy}\right)\left(\frac{dM}{dq} - n\frac{dM}{dp}\right) + C\frac{dM}{dp} + (p + n'q)\frac{dM}{dz} + n'\frac{dM}{dy} + \frac{dM}{dx} = 0.$$

As M must be free from the indeterminate quantities $\frac{dq}{dx}$, $\frac{dq}{dy}$, we must have

$$\frac{dM}{dq} - n\frac{dM}{dp} = 0,$$

$$C\frac{dM}{dp} + (p + n'q)\frac{dM}{dz} + n'\frac{dM}{dy} + \frac{dM}{dx} = 0.$$

We shall have two equations exactly like these for the determination of N. Any two particular solutions which satisfy both these equations may be taken for M and N.

Or assume $dp + ndq - Cdx = 0$, $dy - n'dx = 0$; and with these eliminate C, n, and n' from the above. There results

$$\frac{dM}{dp}dp + \frac{dM}{dq}dq + \frac{dM}{dz}dz + \frac{dM}{dy}dy + \frac{dM}{dx}dx = dM = 0.$$

In like manner we find $dN = 0$. These last therefore are simultaneous with the assumed equations; from which, if any how two complete differentials can be obtained, they are to be taken for $dM = 0$, $dN = 0$. And the integral of the proposed will be $M = \phi(N)$.

We will now take an equation of four variables.

Let $$\frac{d^2z}{dx^2}+A\frac{d^2z}{dy^2}+B\frac{d^2z}{du^2}+C\frac{d^2z}{dx\,dy}+D\frac{d^2z}{dx\,du}+E\frac{d^2z}{dy\,du}=F;$$

$$\frac{dz}{dx}=p,\quad \frac{dz}{dy}=q,\quad \frac{dz}{du}=r.$$

Then $$dp=\frac{d^2z}{dx^2}dx+\frac{d^2z}{dx\,dy}dy+\frac{d^2z}{dx\,du}du,$$

$$dq=\frac{d^2z}{dx\,dy}dx+\frac{d^2z}{dy^2}dy+\frac{d^2z}{dy\,du}du,$$

$$dr=\frac{d^2z}{dx\,du}dx+\frac{d^2z}{dy\,du}dy+\frac{d^2z}{du^2}du.$$

Adding all these, after multiplying the second by m, and the third by n, and subtracting the given equation multiplied by dx, we have

$$dp+mdq+ndr-Fdx=\frac{d^2z}{dy^2}(mdy-Adx)+\frac{d^2z}{du^2}(ndu-Bdx)$$

$$+\frac{d^2z}{dx\,dy}\{dy-(C-m)\,dx\}+\frac{d^2z}{dx\,du}\{du-(D-n)\,dx\}$$

$$+\frac{d^2z}{dy\,du}(mdu+ndy-Edx).$$

Make $$C-m=\frac{A}{m},\quad\text{or } m^2-Cm+A=0;$$

$$D-n=\frac{B}{n},\quad\text{or } n^2-Dn+B=0;$$

and let m, m' be the roots of the former, n, n' those of the latter; also make

$$m\frac{d^2z}{dy^2}+\frac{d^2z}{dx\,dy}=P,\quad n\frac{d^2z}{du^2}+\frac{d^2z}{dx\,du}=Q.$$

Then our equation becomes

$$dp+mdq+ndr-Fdx=P(dy-m'dx)+Q(du-n'dx)$$

$$+\frac{d^2z}{dy\,du}(mdu+ndy-Edx).$$

Or if $$P+n\frac{d^2z}{dy\,du}=P',\quad Q+m\frac{d^2z}{dy\,du}=Q',$$

$$dp+mdq+ndr-Fdx=P'(dy-m'dx)+Q'(du-n'dx)$$

$$+\frac{d^2z}{dy\,du}(mn'+m'n-E)\,dx.$$

This cannot be integrated unless $mn' + m'n - \mathrm{E} = 0$, which is an equation of condition. If this be satisfied, then

$$dp + mdq + ndr - \mathrm{F}dx = \mathrm{P}'\,(dy - m'dx) + \mathrm{Q}'\,(du - n'dx);$$

which may be integrated as an equation of the first order.

Assume $\mathrm{M} = \phi\,(\mathrm{N}, \mathrm{T})$, or $d\mathrm{M} = \omega d\mathrm{N} + \omega' d\mathrm{T}$. Then, as in the case of these variables,

$$\frac{d\mathrm{M}}{dp}\frac{dp}{dx} + \frac{d\mathrm{M}}{dq}\frac{dq}{dx} + \frac{d\mathrm{M}}{dr}\frac{dr}{dx} + \frac{d\mathrm{M}}{dz}p + \frac{d\mathrm{M}}{dx} = 0,$$

$$\frac{d\mathrm{M}}{dp}\frac{dp}{dy} + \frac{d\mathrm{M}}{dq}\frac{dq}{dy} + \frac{d\mathrm{M}}{dr}\frac{dr}{dy} + \frac{d\mathrm{M}}{dz}q + \frac{d\mathrm{M}}{dy} = 0,$$

$$\frac{d\mathrm{M}}{dp}\frac{dp}{du} + \frac{d\mathrm{M}}{dq}\frac{dq}{du} + \frac{d\mathrm{M}}{dr}\frac{dr}{du} + \frac{d\mathrm{M}}{dz}r + \frac{d\mathrm{M}}{du} = 0.$$

Multiply the second by m', the third by n', and add the two products to the first. Subtract the given equation multiplied by $\frac{d\mathrm{M}}{dp}$ from the sum, in order to eliminate $\frac{dp}{dx}$. Thus we have in virtue of the equations

$$m^2 - \mathrm{C}m + \mathrm{A} = 0, \quad n^2 - \mathrm{D}n + \mathrm{B} = 0;$$

$$\left(\frac{dq}{dx} + m'\frac{dq}{dy}\right)\left(\frac{d\mathrm{M}}{dq} - m\frac{d\mathrm{M}}{dp}\right) + \left(\frac{dr}{dx} + n'\frac{dr}{du}\right)\left(\frac{d\mathrm{M}}{dr} - n\frac{d\mathrm{M}}{dp}\right)$$

$$+ \frac{dq}{du}\left(n'\frac{d\mathrm{M}}{dq} + m'\frac{d\mathrm{M}}{dr} - \mathrm{E}\frac{d\mathrm{M}}{dp}\right) + \mathrm{F}\frac{d\mathrm{M}}{dp} + (p + m'q + n'r)\frac{d\mathrm{M}}{dz}$$

$$+ \frac{d\mathrm{M}}{dx} + m'\frac{d\mathrm{M}}{dy} + n'\frac{d\mathrm{M}}{du} = 0.$$

On account of the indeterminates remaining, we must have

$$\mathrm{F}\frac{d\mathrm{M}}{dp} + (p + m'q + n'r)\frac{d\mathrm{M}}{dz} + \frac{d\mathrm{M}}{dx} + m'\frac{d\mathrm{M}}{dy} + n'\frac{d\mathrm{M}}{du} = 0,$$

$$\frac{d\mathrm{M}}{dq} - m\frac{d\mathrm{M}}{dp} = 0, \quad \frac{d\mathrm{M}}{dr} - n\frac{d\mathrm{M}}{dp} = 0,$$

$$n'\frac{d\mathrm{M}}{dq} + m'\frac{d\mathrm{M}}{dr} - \mathrm{E}\frac{d\mathrm{M}}{dp} = 0.$$

The last by elimination gives $mn' + m'n - \mathrm{E} = 0$, which is the equation of condition before found. We shall have equations exactly like them for the determination of N and T.

Now, if we suppose

$$dq + mdp + ndr - \mathrm{F}dx = 0,$$

$dy - m'dx = 0$, $du - n'dx = 0$; and by means of these eliminate F, m, m', n, and n' from the preceding; we again find $d\mathrm{M} = 0$, and consequently $d\mathrm{N} = 0$, $d\mathrm{T} = 0$. These last therefore exist

simultaneously with the assumed equations; from which if we can any how obtain three integrals $M = a$, $N = b$, $T = c$, we shall have, for the integral of the proposed,

$$M = \varphi(N, T).$$

It is further to be observed that the quantities m and m', as also n and n' may change places, if by this means we can obtain other integrals of the first order.

We will next take an equation of the third order. Let

$$\frac{d^3z}{dx^3} + A\frac{d^3z}{dx^2\,dy} + B\frac{d^3z}{dx\,dy^2} + C\frac{d^3z}{dy^3} = D;$$

$$\text{and let } \frac{d^2z}{dx^2} = p', \quad \frac{d^2z}{dx\,dy} = q', \quad \frac{d^2z}{dy^2} = r'.$$

$$\text{Then } dp' = \frac{d^3z}{dx^3}dx + \frac{d^3z}{dx^2\,dy}dy, \quad dq' = \frac{d^3z}{dx^2\,dy}dx + \frac{d^3z}{dx\,dy^2}dy,$$

$$dr' = \frac{d^3z}{dx\,dy^2}dx + \frac{d^3z}{dy^3}dy.$$

Adding all these, after multiplying the second by m and the third by n, and eliminating $\frac{d^3z}{dx^3}$ by the given equation; we obtain

$$dp' + mdq' + ndr' - Ddx = \frac{d^3z}{dx^2\,dy}\{dy - (A-m)dx\}$$

$$+ \frac{d^3z}{dx\,dy^2}\{mdy + (n-B)\,dx\} + \frac{d^3z}{dy^3}(ndy - Cdx).$$

Make $A - m = \frac{C}{n}$, $\frac{B-n}{m} = \frac{C}{n}$. By eliminating m, we find

$$n^3 - Bn^2 + ACn - C^2 = 0.$$

Let the roots of this be n, n', and n''; and let m, m', and m'' be the corresponding values of m. Our equation now becomes

$$dp' + mdq' + ndr' - Ddx = Q\left(dy - \frac{C}{n}dx\right);$$

where Q is a function of $\frac{d^3z}{dx^2\,dy}$, &c., and the differentials are only of the first degree.

Suppose $M = \varphi(N)$ the integral of this. As before, we shall have the partial differentials of each member separately equal to nothing. Or

$$\frac{dM}{dp'}\frac{dp'}{dx} + \frac{dM}{dq'}\frac{dq'}{dx} + \frac{dM}{dr'}\frac{dr'}{dx} + \frac{dM}{dp}\frac{dp}{dx} + \frac{dM}{dq}\frac{dq}{dx} + \frac{dM}{dz}p + \frac{dM}{dx} = 0,$$

$$\frac{dM}{dp'}\frac{dp'}{dy} + \frac{dM}{dq'}\frac{dq'}{dy} + \frac{dM}{dr'}\frac{dr'}{dy} + \frac{dM}{dp}\frac{dp}{dy} + \frac{dM}{dq}\frac{dq}{dy} + \frac{dM}{dz}q + \frac{dM}{dy} = 0.$$

If we eliminate $\frac{dp'}{dx}$ from the first by the given equation, and add the result to the second multiplied by $\frac{C}{n}$; we obtain

$$\left(\frac{dq'}{dx}+\frac{C}{n}\frac{dq'}{dy}\right)\left(\frac{dM}{dq'}-m\frac{dM}{dp'}\right)+\left(\frac{dr'}{dx}+\frac{C}{n}\frac{dr'}{dy}\right)\left(\frac{dM}{dr'}-n\frac{dM}{dp'}\right)$$
$$+\left(p'+\frac{C}{n}q'\right)\frac{dM}{dp}+\left(q'+\frac{C}{n}r'\right)\frac{dM}{dq}+D\frac{dM}{dp'}+\left(p+\frac{C}{n}q\right)\frac{dM}{dz}$$
$$+\frac{dM}{dx}+\frac{C}{n}\frac{dM}{dy}=0.$$

Whence, on account of the indeterminate quantities remaining, we must have

$$\frac{dM}{dq'}-m\frac{dM}{dp'}=0,\quad \frac{dM}{dr'}-n\frac{dM}{dp'}=0,$$
$$\left(p'+\frac{C}{n}q'\right)\frac{dM}{dp}+\left(q'+\frac{C}{n}r'\right)\frac{dM}{dq}+D\frac{dM}{dp'}+\left(p+\frac{C}{n}q\right)\frac{dM}{dz}$$
$$+\frac{dM}{dx}+\frac{C}{n}\frac{dM}{dy}=0.$$

Now assume $dp'+mdq'+ndr'-Ddx=0$, $dy-\frac{C}{n}dx=0$: and by these eliminate C, D, m and n from the preceding equations, and we again find $dM=0$, and consequently also $dN=0$. If then from the assumed equations we can find two integrals $M=a$, $N=b$: we shall have $M=\phi(N)$ for an integral of the proposed of the second order. The two other values of m and n will give two other integrals $M'=\phi(N')$, $M''=\chi(N'')$, when it is practicable to find all three.

In the preceding theory the differentials never rise above the first degree, and they ought not. The method in fact is similar to that employed upon equations of the first order; and it must be evident that it is the natural and proper one. Those who first treated on this subject must have overlooked it from not proceeding in the most simple manner, or from not perceiving how they could eliminate a sufficient number of the indeterminates or differential coefficients.

Denby, near Wakefield.

V.—ON THE EVALUATION OF A DEFINITE MULTIPLE INTEGRAL.

By D. F. GREGORY, B.A., Fellow of Trinity College.

IN a memoir read before the Academy of Sciences of Paris, and inserted in the *Comptes rendus*, Vol. VIII. p. 156, M. Lejeune Dirichlet called the attention of mathematicians to the remarkable multiple integral

$$V = \int dx \int dy \int dz \ldots\ldots x^{a-1}\, y^{b-1}\, z^{c-1} \ldots\ldots (1),$$

which is to be taken between the positive limits of the variables determined by the inequality

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots \overline{\underset{<}{}} 1 \ldots\ldots\ldots (2),$$

the number of variables, and therefore of integrals, being any whatever. The result at which M. Dirichlet arrives is, that

$$V = \frac{\alpha^a \beta^b \gamma^c \ldots}{pqr\ldots} \, \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) \Gamma\left(\frac{c}{r}\right) \ldots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} \ldots\right)},$$

Γ being the second Eulerian Integral.

The actual calculation is not given in the paper referred to, though the process is indicated; but M. Liouville has investigated the value of the integral by a method different from that employed by M. Dirichlet, and his memoir (*Journal de Mathématiques*, Vol. IV. p. 225) is a very elegant specimen of analysis. The integral itself deserves attention, not only as being a remarkable analytical extension of that property of the first Eulerian Integral by which it is connected with the second, but also because it frequently occurs in the investigation of areas of curves, contents of solids, centres of gravity, and other physical and geometrical problems of a similar kind. From its extensive application to cases which are of such frequent occurrence, this multiple integral ought to receive a prominent place in elementary works on the Integral Calculus, and on that account I here bring it before the English reader. The method by which I propose to evaluate this integral is, I believe, new; and I am anxious to show its application in this case, not only because it exhibits very distinctly the nature of the connexion between this integral and the function Γ, but because it can also be applied with great advantage to the calculation of a number of other definite integrals. In the present paper, however, I shall confine myself to the integral of M. Dirichlet, and a more general one of the same kind which is given by M. Liouville.

In the first place, following the method of M. Liouville, we shall

transform the integral so that the limits shall be of the first degree only. This is easily done by assuming

$$\left(\frac{x}{\alpha}\right)^p = x', \quad \left(\frac{y}{\beta}\right)^q = y', \quad \left(\frac{z}{\gamma}\right)^r = z', \text{ \&c.}$$

from which

$$dx = \frac{\alpha}{p} x'^{\frac{1}{p}-1}\, dx', \quad dy = \frac{\beta}{q} y'^{\frac{1}{q}-1}\, dy', \quad dz = \frac{\gamma}{r} z'^{\frac{1}{r}-1}\, dz', \text{ \&c.}$$

On substituting these values for the variables and their differentials, the integral becomes

$$V = \frac{\alpha^a \beta^b \gamma^c \ldots}{pqr \ldots} U \ldots\ldots\ldots (3),$$

where U is the definite integral

$$\int dx' \int dy' \int dz' \ldots x'^{\frac{a}{p}-1} y'^{\frac{b}{q}-1} z'^{\frac{c}{r}-1} \ldots (4),$$

the limits of the variables being given by the inequality

$$x' + y' + z' \ldots\ldots \overline{\overline{<}}\, 1 \ldots\ldots\ldots\ldots\ldots (5).$$

Now let

$$\frac{a}{p} = l, \quad \frac{b}{q} = m, \quad \frac{c}{r} = n \ldots\ldots ..$$

and, dropping the accents which are no longer necessary for discrimination, we have to calculate the integral

$$U = \int dx \int dy \int dz \ldots\ldots x^{l-1} y^{m-1} z^{n-1} \ldots\ldots (6),$$

the limits being given by the inequality

$$x + y + z \ldots\ldots \overline{\overline{<}}\, 1.$$

If the variables be only two in number, x and y, the integral is reduced to

$$U = \int dx \int dy\, x^{l-1} y^{m-1} = \frac{1}{m} \int dx\, x^{l-1} (1 - x)^m,$$

since the limits of y are 0 and $1 - x$.

The evaluation of this integral, by a method due to Professor Jacobi, may be found in this Journal, Vol. I., p. 94, and it is by an extension of that method that M. Liouville has calculated the general integral under consideration. Instead of employing it we shall proceed in the following manner.

Let $$x + y + z + \ldots\ldots\ldots = v,$$
or $$x = v - y - z - \ldots\ldots$$

Then, as x varies when y, z ..... are constant, $dx = dv$, and (10) becomes

$$U = \int dv \int dy \int dz\, y^{m-1} z^{n-1} (v - y - z \ldots)^{l-1} \ldots\ldots\ldots (7).$$

We might now integrate with respect to v but, for the conveni-

ence of our future operations, we shall only indicate the operations. The extreme limits of v are 0 and 1, and we may therefore write

$$U=\int_0^1 dv\int dy\int dz\, y^{m-1}z^{n-1}(v-y-z-\ldots)^{l-1}.$$

Now by the symbolical form of Taylor's Theorem we have

$$f(x+h) = \varepsilon^{h\frac{d}{dx}}f(x).$$

Hence we may put

$$(v-y-z-\ldots)^{l-1} = \varepsilon^{-y\frac{d}{dv}}(v-z-\ldots)^{l-1} \ldots\ldots (9),$$

and we have then

$$U=\int_0^1 dv\int dz\, z^{n-1}\ldots\int dy\, y^{m-1}\varepsilon^{-y\frac{d}{dv}}(v-z-\ldots)^{l-1}\ldots\ldots(10),$$

the limits of y being 0 and $v - z - \ldots$.

Now assume $y\dfrac{d}{dv} = t$, so that $dy = dt\left(\dfrac{d}{dv}\right)^{-1}$,

$$U=\int_0^1 dv\int dz\, z^{n-1}\int dt\, t^{m-1}\varepsilon^{-t}\left(\frac{d}{dv}\right)^{-m}(v-z\ldots)^{l-1}\ldots(11).$$

To find the limits of t we have recourse to the following considerations. Supposing, for simplicity, that there were only two variables, x and y, we have

$$(v-y)^{l-1} = \varepsilon^{-y\frac{d}{dv}}v^{l-1} = \varepsilon^{-t}.\,v^{l-1}.$$

Now, when $y = 0$ the first side becomes v^{l-1}; and in order that the second side should be reduced to that form, we must have $t = 0$. Again, when $y = v$ the first side becomes zero, l being positive; and in order that the second side may also become zero, we must have $t = \infty$.

Hence to the values

$$\left.\begin{matrix} y = 0 \\ y = v \end{matrix}\right\} \text{ correspond } \left\{\begin{matrix} t = 0 \\ t = \infty. \end{matrix}\right.$$

As in this transformation consists the principle of the method I employ, I shall add a few words in explanation of it. When we assume

$$y\frac{d}{dv} = t \text{ and } dy = dt\left(\frac{d}{dv}\right)^{-1},$$

and therefore

$$\varepsilon^{-y\frac{d}{dv}}v^{l-1} = \varepsilon^{-t}v^{l-1},$$

we suppose t to be a variable capable of increase or decrease; or, in other words, a symbol of quantity. But, on the other hand, $y\dfrac{d}{dv}$ is a symbol of operation to which we cannot apply the terms increase or decrease, and it may therefore seem to be scarcely allowable to assume it to be equal to a quantitative symbol It is to be

observed, however, that our use of the symbolical expression is only for the assumption of the *form* of the new function, and that when we put $y\dfrac{d}{dv}=t$, we really say nothing more than that t is to be such a function of y that it shall satisfy the equation

$$(v-y)^{l-1}=\varepsilon^{-t}v^{l-1},$$

which of course is an assumption which we are quite at liberty to make. Whether in every case of such a transformation we could actually determine t, is a question with which we have fortunately no concern, as it is to be expected that such a determination would be in most cases very difficult if not impracticable. All that we require to know are the limiting values of t corresponding to those of y; and as these can generally be found by considerations such as we have used, the transformation is for our purposes sufficient. It reduces the function to be integrated to a very convenient form for effecting that operation, and the substitution of the values of the limits offers generally no difficulties. What we have said regarding the limits in the case of two variables, applies equally well to a greater number, except that the limits of y being 0 and $v-z-\ldots$, the limits of t will still be 0 and ∞. Hence, recurring to our integral, equation (11) becomes, on affixing the limits to t,

$$\mathrm{U}=\int_0^1 dv\int dz\, z^{n-1}\ldots\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}.$$

Now $\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}=\Gamma(m)$, and therefore

$$\mathrm{U}=\Gamma(m)\int_0^1 dv\int dz\, z^{n-1}\ldots\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}\;\ldots\ldots(12).$$

Proceeding with z in the same manner as with y, by putting

$$(v-z-\ldots)^{l-1}=\varepsilon^{-z\frac{d}{dv}}v^{l-1},$$

and assuming

$$z\frac{d}{dv}=s,$$

we find, as before,

$$\mathrm{U}=\Gamma(m)\int_0^1 dv\ldots\int_0^\infty ds\, s^{n-1}\varepsilon^{-s}\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1},$$

$$=\Gamma(m)\,\Gamma(n)\int_0^1 dv\ldots\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}\ldots\ldots(13).$$

In this manner we might proceed for any number of variables, but restricting ourselves to three, it only remains to integrate with respect to v between the given limits.

Now as

$$\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}=\frac{\Gamma(l)}{\Gamma(l+m+n)}v^{l+m+n-1},$$

we find

$$\begin{aligned}U &= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_0^1 dv\, v^{l+m+n-1} \\ &= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{(l+m+n)\,\Gamma(l+m+n)} \\ &= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+l+m+n)} \quad \ldots\ldots\ \ldots\ldots\ldots (14);\end{aligned}$$

since by the fundamental property of the function Γ

$$(l+m+n)\,\Gamma(l+m+n) = \Gamma(1+l+m+n).$$

Substituting this in (3) and putting for l, m, n, their values $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$, we finally obtain

$$V = \frac{\alpha^a \beta^b \gamma^c}{pqr} \, \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)}{\Gamma\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)} \quad \ldots.\ \ldots(15).$$

It is to be observed, that in effecting the operation

$$\left(\frac{d}{dv}\right)^{-(m+n)} v^{l-1},$$

we must not add any arbitrary constants, since this inverse operation has arisen during the process without causing any constants to disappear, and there are therefore none to be restored. All the arbitrary constants arising from the original integrals are eliminated in taking the limits, and no others are to be introduced, otherwise we should have more arbitrary constants than integrals.

The transformation in (11) and the subsequent investigation of the limits are the parts of this method which appear to be new, or at least not to have been hitherto employed to calculate definite integrals. It is easily seen that the same transformation may be applied to many other definite integrals, but it would occupy too much space to enter on the consideration of them at present. I shall therefore pass on to some examples of the application of the formula which has just been proved.

Ex. 1. To find the area of the evolute to the ellipse.

The expression for the area is

$$V = \iint dx\, dy,$$

x and y being subject to the limiting condition

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

Here $a = 1$, $b = 1$, $p = q = \frac{2}{3}$. Therefore by (15)

$$V = \frac{9}{4}\,\alpha\beta\,\frac{\{\Gamma(\frac{3}{2})\}^2}{\Gamma(4)}.$$

Now $\Gamma(4)=3.2.1$, and $\Gamma(\frac{3}{2})=\frac{1}{2}\,\Gamma(\frac{1}{2})=\frac{1}{2}\sqrt{\pi}$,

and therefore $V=\dfrac{3\alpha\beta}{32}$, which is the area of the portion of the curve included between the positive axes, since the variables are supposed never to become negative. The area of the whole curve is $\dfrac{3\alpha\beta}{8}$.

Ex. 2. If we wish to find the co-ordinates of the centre of gravity of the same area, we have to calculate

$$\iint x\,dx\,dy \text{ and } \iint y\,dx\,dy.$$

By the formula (15)

$$\iint x\,dx\,dy=\frac{9}{4}\alpha^2\beta\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})},\ \iint y\,dx\,dy=\frac{9}{4}\alpha\beta^2\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})}.$$

Hence if $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity,

$$\bar{x}=\frac{2^8\alpha}{9.7.5},\quad \bar{y}=\frac{2^8\beta}{9.7.5}.$$

Ex. 3. It is shown in this Journal (Vol. II p. 14,) that the equation to the parabola, when referred to two tangents as axes, is

$$\sqrt{\frac{x}{\alpha}}+\sqrt{\frac{y}{\beta}}=1,$$

α and β being the portion of the tangents intercepted between their intersection and the curve. If θ be the angle between the axes, the area is

$$V=\sin\theta\iint dx\,dy,$$

the limits of x and y being given by the preceding equation. Here $a=1$, $b=1$, $p=q=\frac{1}{2}$. Therefore

$$V=\sin\theta\,.\,4\alpha\beta\,\frac{\{\Gamma(2)\}^2}{\Gamma(5)}=\frac{\alpha\beta\sin\theta}{6}.$$

From this it appears (referring to the figure in the article alluded to above,) that the triangle ABC is three times the area ABPC.

Ex. 4. To find the centre of gravity of the eighth part of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

If $\bar{z}$ be one of the co-ordinates of the centre of gravity,

$$\bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

the limits being given by the preceding equation.

In the numerator, comparing it with (15), we have

$$a=b=1,\quad c=2,\quad p=q=r=2.$$

Therefore

$$\iiint z\,dx\,dy\,dz = \frac{\alpha\beta\gamma^2}{8}\,\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(3)} = \pi\frac{\alpha\beta\gamma^2}{16}.$$

In the denominator $a=b=c=1$, $p=q=r=2$. Therefore

$$\iiint dx\,dy\,dz = \frac{\pi\alpha\beta\gamma}{8}\,\frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(1+\frac{3}{2})} = \frac{\pi\alpha\beta\gamma}{6}.$$

Hence $\bar{z} = \frac{3}{8}\gamma$, and similarly for the other co-ordinates.

M. Liouville has given to the Theorem of M. Dirichlet a very important extension, of which the following is the enunciation. If

$$W = \int dx \int dy \int dz \ldots x^{l-1} y^{m-1} z^{n-1} \ldots f(x+y+z+\ldots) \quad (16),$$

where the limits of $x, y, z\ldots$ are such as to satisfy the inequality

$$x+y+z+\ldots \overline{\overline{<}}\, h,$$

f being any function whatsoever,

$$W = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(l+m+n+\ldots)} \int_0^h dv\, f(v)\, v^{l+m+n+\ldots-1} \ldots\ldots (17).$$

It will be seen, as in the first part of this article, that to the form (16) may be reduced the more general one

$$W' = \int dx \int dy \int dz \ldots x^{a-1}\, y^{b-1}\, z^{c-1} f\left\{\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\right\},$$

where the limiting values are given by the inequality

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\ldots \overline{\overline{<}}\, h;$$

for by a simple transformation of the variables we should find

$$W' = \frac{\alpha^a \beta^b \gamma^c}{pqr}\, W,$$

and it is therefore only necessary to calculate W. To effect this we proceed in the same manner as before.

Let $\qquad x + y + z + \ldots = v,$

then $\quad dx = dv,$

the extreme limits of v being 0 and h. Then

$$W = \int_0^h dv \int dy \int dz \ldots y^{m-1}\, z^{n-1} \ldots (v-y-z-\ldots)^{l-1} f(v).$$

Now, as before,

$$(v-y-z-\ldots)^{l-1} f(v) = \epsilon^{-y\frac{d'}{dv}} (v-z-\ldots)^{l-1} f(v),$$

where $\frac{d'}{dv}$ is accentuated to imply that it refers only to the v included in $(v-z-\ldots)^{l-1}$ and not to that under the f. Now putting $y\frac{d'}{dv} = t$, the limits of t will be 0 and ∞, and

$$W=\int_0^h dv \int dz \ldots z^{n-1} \ldots \int_0^\infty dt\, t^{m-1}\, \epsilon^{-t} \left(\frac{d}{dv}\right)^{-m} (v-z-\ldots)^{l-1} f(v),$$

$$=\Gamma(m) \int_0^h dv \int dz \ldots z^{n-1} \ldots \left(\frac{d}{dv}\right)^{-m} (v-z-\ldots)^{l-1} f(v).$$

Next, treating z in the same manner as y, we have

$$W=\Gamma(m)\,\Gamma(n) \int_0^h dv \ldots \left(\frac{d}{dv}\right)^{-(m+n)} (v-\ldots)^{l-1} f(v),$$

and so on for any number of variables. Restricting ourselves to three, and effecting the operation $\left(\frac{d}{dv}\right)^{-(m+n)}$, which has reference only to v^{l-1}, we find

$$W=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_0^h dv\, v^{l+m+n-1} f(v).$$

As an example of the application of this formula, let us take the expression

$$W=\int dx \int dy \int dz \frac{1}{\sqrt{1-x^2-y^2-z^2}},$$

where the variables satisfy the condition

$$x^2+y^2+z^2 \leqq 1.$$

To change the variables, put $x^2=x'$, $y^2=y'$, $z^2=z'$; then

$$W=\tfrac{1}{8}\iiint \frac{dx'\, dy'\, dz'}{\sqrt{x'y'z'}} \frac{1}{\sqrt{1-x'-y'-z'}},$$

$$\text{and } x'+y'+z' \leqq 1.$$

In this case, then, $l=m=n=\frac{1}{2}$; therefore

$$W=\frac{1}{8}\frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(\frac{3}{2})} \int_0^1 \frac{v^{\frac{1}{2}} dv}{\sqrt{1-v}};$$

putting $v=x^2$, we find

$$\int_0^1 \frac{v^{\frac{1}{2}} dv}{\sqrt{1-v}} = 2\int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\pi.$$

Also $\Gamma(\frac{1}{2})=\sqrt{\pi}$, and $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}$; therefore

$$W=\frac{\pi^2}{8}.$$

Again, take $W=\iint dx\, dy \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}$,

when $x^2+y^2 \leqq 1$.

By a change of the variable,

$$W=\tfrac{1}{4}\iint \frac{dx\, dy}{\sqrt{xy}} \sqrt{\frac{1-x-y}{1+x+y}}, \text{ and } x+y \leqq 1.$$

Here $l=m=\frac{1}{2}$; therefore

$$W=\tfrac{1}{4}\,\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)}\int_0^1 dv\,\sqrt{\frac{1-v}{1+v}}.$$

Now $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$,

$$\int_0^1 dv\,\sqrt{\frac{1-v}{1+v}}=\int_0^1 dv\left(\frac{1}{\sqrt{1-v^2}}-\frac{v}{\sqrt{1-v^2}}\right)=\frac{\pi}{2}-1;$$

$$\text{therefore}\quad W=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$$

Other examples of this formula will be found in a paper by M. E. Catalan, *Journal de Mathématiques*, Vol. IV. p. 323.

VI.—REMARKS ON POISSON'S PROOF OF THE PROPOSITION THAT $F(\mu.\omega)$ MAY BE EXPANDED IN A SERIES OF LAPLACE'S COEFFICIENTS.*

I PROPOSE in the following paper to simplify the integration which occurs in the proof of this proposition as given by Poisson (*Théorie de la Chaleur*, Art. 106), and by Pratt (*Mechanics*, Art. 176), and to allude to a point which appears to present some difficulty in the articles just referred to.

The definite integral which occurs is

$$\int_{-1}^{1}\int_0^{2\pi}\frac{(1-c^2)\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}}\ (=\text{X suppose}),$$

$\{$where $p=\mu\mu'+\sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\omega-\omega')\}$ when $c=1$.

It is manifest that the element of the integral will have no sensible value unless $p=1$; that is, unless $\mu'=\mu$ and $\omega'=\omega+$ some multiple of 2π. Let then the integral be transformed, as in Pratt, and we have

$$X=\iint\frac{2g\,d\nu\,dz}{\left\{g^2+\dfrac{\nu^2}{1-\mu^2}+z^2(1-\mu^2)\right\}^{\frac{3}{2}}}$$

$$=\iint\frac{2g\,d\nu'\,dz'}{(g^2+\nu'^2+z'^2)^{\frac{3}{2}}},$$

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* From a Correspondent.

$$\text{if } \nu' = \frac{\nu}{\sqrt{1-\mu^2}}, \quad z' = z\sqrt{1-\mu^2}.$$

Now for distinctness take the rectangle $acdb$, (fig. 2),

$$Aa = Ac = 1, \quad AB = 2\pi.$$

Take A as origin, and let values of ω' be represented by lines parallel to AB, values of μ' by lines perpendicular to it. Then, since the limits of μ' are $+1$ and -1, and those of ω' 2π and 0, all points whose co-ordinates are corresponding values of ω' and μ' will lie within $acdb$. And if we take P as a point represented by the values of ω and μ, the above integration will be effected by integrating through any small space round P, since it is only at P that the elements are not evanescent.

These considerations of space suggest a simplification of the integral, viz. by trausforming it to polar co-ordinates r and θ: which being done, we have

$$X = \iint \frac{2grd\theta dr}{(g^2+r^2)^{\frac{3}{2}}}.$$

Now suppose,

I. That μ and ω have not the values ± 1 or 2π and 0, so that P lies within $acdb$. Then we must integrate through a small space round P, and the limits will therefore be 0 and 2π for θ, and 0 and any small value (a) for r,

$$X = 4\pi g \int \frac{rdr}{(g^2+r^2)^{\frac{3}{2}}} = 4\pi g \left\{\frac{1}{g} - \frac{1}{\sqrt{g^2+a^2}}\right\}$$

$$= 4\pi \left\{1 - \frac{1}{\sqrt{1+\frac{a^2}{g^2}}}\right\}.$$

Put $g = 0$; then, since a is finite, $\frac{a}{g} = \infty$, and therefore $X = 4\pi$.

There is a difficulty at this point in Poisson's analysis. He says, that he proposes taking $+\infty$ and $-\infty$ as the limits of integration for $z-$ limits; of which, to say the least, the admissibility ought to be proved,—because it is not necessary, in order that the elements of the integral should be sensible, that ω' should be indefinitely nearly equal to ω, but only that it should either be equal or differ from it by some even multiple of π. And hence if $+\infty$ and $-\infty$ be the limits of z, it seems hard to assert, *à priori*, that no elements will be included in the integral but such as properly belong to it. But Poisson does not, in fact, take the above limits, for he transforms the integral by putting

$$z^2(1-\mu^2) = \left(g^2 + \frac{\nu^2}{1-\mu^2}\right)x^2;$$

and then takes $\pm\infty$ as the limits of x. Now it will be observed, that x is multiplied by an evanescent quantity, and therefore it does not follow that z should be infinite because x is so.

The truth of these remarks will, I think, appear from observing the method above given of finding the value of X. We find that

$$X = 4\pi \left\{1 - \frac{1}{\sqrt{1 + \frac{a^2}{g^2}}}\right\}.$$

The assumption of $\pm\infty$ for the limits of z, will be somewhat similar in its nature to that of 0 and ∞ for those of r; this will be effected by putting $a = \infty$, in which case we shall, it is true, get the same value of X as before; but the assumption is not necessary, since if a be finite, $\frac{a}{g}$ will be infinite, which is all we require. Moreover, if $a = \infty$, we should have $X = 4\pi$, independently of the value of g; whereas the evanescence of g is an essential condition of the problem.

II. The case in which μ has one of the values $+1$ or -1, or ω one of the values 2π or 0, will require a few words.

1. Suppose $\omega = 0$, then referring to the figure there will be *two* points Q, Q′, for which $\omega' = 0$, and $\omega' = 2\pi$, for which the elements of the integral will not vanish.

The limits of r may be the same as before, but those of θ must be $\pm\frac{\pi}{2}$; since otherwise we should integrate out of the rectangle *acbd*. The value of the integral for each point will be 2π, and therefore on the whole $X = 4\pi$, as before.

The same remarks will apply when $\omega = 2\pi$.

2. Suppose $\mu = 1$. Then there will be one point, as R, (see fig. 2) on the circumference of *acdb*, for which the value of the elements will be sensible. It would appear at first sight that the limits of θ should be 0 and π in this case, and thus we should have $X = 2\pi$; that this is not the case, may be easily shewn by an independent investigation, as in Pratt (page 168), and the reason of this may perhaps be given thus: we have assumed

$$\nu' = \frac{\nu}{\sqrt{1-\mu^2}},$$

$$\text{and} \quad \nu' = r \sin\theta;$$

$$\therefore \nu = \sqrt{1-\mu^2}\,.\, r \sin\theta.$$

In the particular case, therefore, of $\mu = 1$, we see that $\nu = 0$ whatever be θ; and we shall therefore not go beyond our limits (or integrate outside the rectangle *acdb*), even though we take 0 and 2π as the limits of θ.

The same explanation will apply when $\mu = -1$.

H. G.

VII.—ON THE DEVELOPMENT OF THE SQUARE ROOTS OF INTEGRAL AND FRACTIONAL NUMBERS BY CONTINUED FRACTIONS.

By JAMES BOOTH, M.A., Principal of and Professor of Mathematics in Bristol College.

THAT the square root of a number may be exhibited in the form of a continued fraction, is a property of such fractions long known, yet the proof usually given, showing that the terms of such fraction are periodic, appears both tedious and obscure; and the inverse problem, to determine the numbers whose square roots may be developed in periods of one, two, three, or more given terms, appears never to have attracted the attention of mathematicians: to discuss this is the object of the present paper.

Assuming as known the common properties of continued fractions, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, be the three final consecutive converging fractions of the period, and let μ be the last quote of the period;

$$\text{then}\quad R = Q\mu + P, \quad R' = Q'\mu + P' \ldots\ldots \ldots (1),$$

α, β, γ, $\delta \ldots \mu$ being the successive quotes of the period.

The rule by which these converging fractions are found is—multiply the numerator of the converging fraction just found by the corresponding quote, and add the numerator of the preceding fraction; this sum will be the numerator of the next converging fraction: and the denominator may be found in a precisely similar manner.

Let N be the number whose square root is required, a^2 a square contained in N, which, when the number is an integer, will be the greatest possible, k the difference between N and a^2, so that

$$N = a^2 + k \ldots \ldots\ldots\ldots (2),$$

$$\text{and let}\quad \sqrt{N} = a + z \ldots\ldots\ldots\ldots (3),$$

$$\text{then}\quad z = \cfrac{1}{\alpha + \cfrac{1}{\beta + \cdots \atop \cdots\cdots \cdots \cfrac{+1}{\mu + z}}} \ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots (4).$$

Now the final converging fraction of the first period is

$$\frac{R}{R'} = \frac{Q\mu + P}{Q'\mu + P'},$$

which involves μ only in the first power; hence, as μ and z are

similarly involved in (4), the complete value of the final converging fraction is

$$\frac{Q(\mu+z)+P}{Q'(\mu+z)+P'};$$

equating this expression with z, we find

$$z=-\left(\frac{R'-Q}{2Q'}\right)+\sqrt{\left(\frac{R'-Q}{2Q'}\right)^2+\frac{R}{Q'}} \;\ldots.\; (5),$$

but from (3) $\qquad z=-a+\sqrt{N}.$

Equating those values of z, comparing the rational and irrational parts together, we find

$$a=\frac{R'-Q}{2Q'} \;\ldots\ldots\ldots\ldots(6),$$

$$N=\left(\frac{R'-Q}{2Q'}\right)^2+\frac{R}{Q'} \;\ldots\ldots\ldots\ldots(7);$$

or, in the last equation, introducing the value of $\left(\frac{R'-Q}{2Q}\right)$ given by (6), we get

$$N=a^2+\frac{R}{Q'} \;\ldots\ldots\ldots\ldots\ldots (8);$$

but by (2),

$$N=a^2+k, \quad \text{or } k=\frac{R}{Q'} \;\ldots\ldots\ldots(9):$$

hence we obtain the following Theorem:

In reducing the square root of a number, whether integral or fractional, to the form of a continued fraction, the numerator of the final converging fraction of a period bears to the denominator of the preceding converging fraction a given ratio.

From equation (6) we find $R'=2aQ'+Q$; but by the general rule for the formation of the converging fractions, $R'=\mu Q'+P'$; hence, eliminating R' we find

$$a=\frac{\mu}{2}+\frac{P'-Q}{2Q'}\;\ldots\ldots\ldots(10).$$

Now when N is an integer a must be so also; and that this may be always the case, P' must be equal to Q: whence it follows that $\mu=2a$, or *when the square root of any integer is developed in the form of a continued fraction, the final quote μ of the period is always double the integral part of the root, and the numerator of the last converging fraction of the period but one is equal to the denominator of the last converging fraction but two.* Hence μ is always an even number, let it be $=2\lambda$.

To find the form of those integral numbers whose square roots may be developed in periods consisting of only one term,

let the quotes be $\quad 0,\ 2\lambda,$

and the converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{2\lambda};$

$$\text{hence}\quad R'=2\lambda,\ R=1,\quad Q'=1,\ Q=0,$$

or $\dfrac{R'-Q}{2Q'}=\lambda$ an integer, and $k=\dfrac{R}{Q'}=1$; therefore

$$N=(\lambda^2+1);$$

or all numbers of the form (λ^2+1) may have their square roots developed under the form of a continued fraction, the period consisting of only one term; thus

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2}+1\ \&c.}\qquad\qquad \sqrt{17}=4+\cfrac{1}{8+\cfrac{1}{8}+1,\ \&c.}$$

To find the form of those integral numbers whose square roots may be developed in periods of two terms,

let the quotes be $\quad 0,\ \alpha,\ 2\lambda,$

and the corresponding converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{\alpha},\ \frac{2\lambda}{2\lambda\alpha+1};$

$$\text{or}\quad R'=2\alpha\lambda+1,\quad R=2\lambda,\quad Q'=\alpha,\quad Q=1;$$

$$\text{hence}\quad \alpha=\frac{R'-Q}{2Q'}=\lambda,\quad \text{and}\ k=\frac{R}{Q'}=\frac{2\lambda}{\alpha}:$$

hence 2λ must be a multiple of α.

Let $\lambda=t\alpha$; then all numbers of the form $(t^2\alpha^2+2t)$, where α and t are any integral numbers whatever, may have their square roots developed in periods of two terms. Thus, let $\alpha=5,\ t=1,$

$$\sqrt{27}=5+\cfrac{1}{5+\cfrac{1}{10+\cfrac{1}{5}}}$$

let $\alpha=3,\ t=2,$

$$\sqrt{40}=6+\cfrac{1}{3+\cfrac{1}{12+\cfrac{1}{3}}}$$

When the period is to consist of three terms, to find the form of the numbers,

let the quotes be $\quad 0,\ \alpha,\ \beta,\ 2\lambda,$

and the converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{\alpha},\ \frac{\beta}{\alpha\beta+1},\ \frac{2\beta\lambda+1}{2\alpha\beta\lambda+2\lambda+\alpha}:$

then, as the number N is an integer, $Q=P'$, or $\alpha=\beta$,

$$\text{and}\quad \frac{R}{Q}=k=\frac{2\alpha\lambda+1}{\alpha^2+1},\ \text{putting}\ \alpha\ \text{for}\ \beta;$$

hence $\frac{2a\lambda+1}{a^2+1}$ must be an integer. Write this fraction in the form

$$\frac{2a\lambda+a^2+1-a^2}{a^2+1}=k, \quad \text{or} \quad \frac{a(2\lambda-a)}{a^2+1}=k-1;$$

hence $(2\lambda-a)$ must be a multiple of (a^2+1), or $2\lambda-a=t(a^2+1)$, or $\lambda=\frac{ta^2+t+a}{2}$; hence both t and a must be even.

$$\text{Let } t=2\tau \text{ and } a=2\delta,$$

$$\text{then } \lambda=4\tau\delta^2+\tau+\delta \text{ and } k=4\tau\delta+1;$$

hence $\quad N=(\lambda^2+k)=(\delta^2+1)+2\delta(4\delta^2+3)\tau+(4\delta^2+1)^2\tau^2.$

Thus, let $\delta=1$, $\tau=1$,

$$\sqrt{41}=6+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{12}+\&c.}}$$

let $\delta=2$, $\tau=2$,

$$\sqrt{1313}=36+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{72}}}$$

When the period consists of four terms, let the quotes and corresponding converging fractions be

$$0,\quad a,\quad \beta,\quad \gamma \qquad 2\lambda$$

$$\frac{1}{0},\ \frac{0}{1},\ \frac{1}{a},\ \frac{\beta}{a\beta+1},\ \frac{\beta\gamma+1}{a\beta\gamma+a+\gamma},\ \frac{2\beta\lambda\gamma+2\lambda+\beta}{2a\beta\gamma\lambda+2a\lambda+2\gamma\lambda+a\beta+1}.$$

Now as the required number is to be integral,

$$Q=P', \text{ or } \beta\gamma+1=a\beta+1, \text{ or } \gamma=a;$$

hence the third term of the period must be equal to the first: and as $\frac{R}{Q'}$ must be an integer $\frac{2\lambda a\beta+2\lambda+\beta}{a^2\beta+2a}$ must be an integer k, putting a for γ.

$$\text{Let } \lambda=\nu a, \quad \text{then } k=2\nu-\frac{2\nu a-\beta}{a^2\beta+2a};$$

and as $\frac{2\nu a-\beta}{a^2\beta+2a}$ must be an integer, let $\beta=\delta a$;

then $\frac{2\nu-\delta}{\delta a^2+2}$ must be an integer $=\eta$;

$$\text{hence} \quad 2\nu=\delta+2\eta+\eta\delta a^2;$$

$$\text{hence} \quad k=\delta+\eta+\delta\eta a^2, \quad \text{and } \lambda=\eta a+\delta a\left(\frac{1+\eta a^2}{2}\right),$$

therefore $N=(\lambda^2+k)=\eta\,(1+\eta a^2)+(4\delta+\delta^2 a^2)\left(\frac{1+\eta a^2}{2}\right)^2$.

Let $a=2$, $\delta=2$, $\eta=1$, then $N=155$; therefore

$$\sqrt{155}=12+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{24}}}}$$

To find the number whose square root may give the quotes 1, 2, 3, 4; the quotes and converging fractions are

$$\begin{array}{cccccc} 0, & 1, & 2, & 3, & 4, & \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{1}, & \frac{2}{3}, & \frac{7}{10}, & \frac{30}{43}. \end{array}$$

Now the integral part of the root is $\frac{R'-Q}{2Q'}=\frac{43-7}{2\,.\,10}=\frac{9}{5}$, and $k=\frac{R}{Q'}=3$; hence $N=a^2+k=\frac{156}{25}$: consequently

$$\sqrt{\frac{156}{25}}=\frac{9}{5}+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{1+\&c.}}}}}$$

The square root of any number being developed in the form of a continued fraction, such fraction will be periodic.

Assume the equations (6), (8),

$$\left.\begin{array}{c} N=a^2+\frac{R}{Q'},\quad R'=2aQ'+Q, \\ \text{and } R'Q-RQ'=\pm 1, \end{array}\right\}\ldots\ldots\ldots(A);$$

the upper sign to be taken when $\frac{R}{R'}$ holds an *even* place in the series of fractions $\frac{1}{0}$, $\frac{0}{1}$, ... $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, and *vice versâ.* From these three equations, eliminating R and R′, we find

$$Q=\sqrt{NQ'^2\pm 1}-aQ'\ \ldots\ldots(11).$$

Now such a value for Q' must be found as will render $(NQ'^2\pm 1)$ a complete square, and greater than $a^2Q'^2$; and as such a value of Q' is always possible, it is manifest that neither the numerator nor denominator of the last converging fraction of the period can ever become infinite: thence, as the numerators and denominators of

these converging fractions, from the law of their formation, are continually increasing in magnitude, it is evident that a finite number of converging fractions must intervene between the first and last of the period; and as a partial quote exists for every such fraction, it follows that the number of such quotes is finite, or the period is finite.

Hence we may derive a singular theorem, namely, that the numbers which, substituted for t, will render the formula $(At^2 \pm 1)$ a perfect square, are the denominators of the penultimate converging fractions in each period of the development of the square root of A.

Thus, let $A=11$; then the converging fractions are

$$\frac{1}{3}, \frac{6}{19}, \frac{19}{60}, \frac{120}{379}, \frac{379}{1197}, \text{\&c.}$$

it will be found that 3, 60, 1197, substituted for t, will render $(11t^2+1)$ a complete square.

By the aid of these principles the converging fractions of a period of the development of the square root of A may be found in an inverse order: thus, let $A=27$, then $\sqrt{(27Q'^2+1)}$ is rational when $Q'=5$, and

$$\sqrt{27 \cdot Q'^2 + 1} = \sqrt{676} = 26,$$

$$Q = \sqrt{NQ'^2 + 1} - aQ' = 26 - 25 = 1,$$

$$\text{and } P' = 1, \quad \text{and } R = kQ' = 10, \quad \text{also } R' = 51;$$

hence the fractions are $\frac{10}{51}, \frac{1}{5}, \frac{1}{1}$.

Had Q and Q′ been eliminated from equations (A), we should have found

$$R = \sqrt{NR'^2 \mp k} - aR' \;\ldots\ldots\ldots\; (12),$$

$$\text{or } \; R = \sqrt{NQ^2 \mp k} + aQ \;\ldots\ldots\ldots\; (13),$$

had we eliminated R′ and Q′.

Hence it follows, that if the square root of a number N be developed in the form of a continued fraction, the denominators of the last converging fractions of each period, or the numerators of the immediately preceding fractions, substituted for t, will render the formula $(Nt^2 \mp k)$ a complete square, k being the difference between N and the greatest square contained in it.

Thus, let $N=7$; here $k=3$, and the converging fractions being

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{9}{14} \,\Big|\, \frac{11}{17}, \frac{20}{31}, \frac{31}{48}, \frac{144}{223} \,\Big|\, \frac{175}{271}, \frac{319}{494}, \frac{494}{765}, \frac{2295}{3554} \,\Big|\, \text{\&c.}$$

it will be found that 2, 31, 494, or 14, 223, 3554, put for t in $(7t^2-3)$, will render this expression a square; or 3, 48, 765, put for t in $(7t^2+1)$, will constitute this formula a complete square.

VIII.—EXAMPLES OF THE DIALYTIC METHOD OF ELIMINATION AS APPLIED TO TERNARY SYSTEMS OF EQUATIONS.

By J. J. Sylvester, F.R.S., Professor of Natural Philosophy in University College, London.

This method is of universal application, and at once enables us to reduce any case of elimination to the form of a problem, where that operation is to be effected between quantities linearly involved in the equations which contain them.

As applied to a binary system, $fx = 0$, $\phi x = 0$, the method furnishes a rule by which we may unfailingly arrive at *the determinant*, free from every species of irrelevancy, whether of a linear, factorial, or numerical kind.

The rule itself is given in the *Philosophical Magazine*, (London and Edinburgh, Dec 1840). The principle of the rule will be found correctly stated by Professor Richelot, of Konigsberg, in a late number of *Crelle's Journal*, at the commencement of a memoir in Latin bordering on the same subject, ("Nota ad Eliminationem pertinens.")

My object at present is to supply a few instances of its application to ternary systems of equations.

Ex. 1. To eliminate x, y, z, between the three homogeneous equations.

$$\left.\begin{aligned} Ay^2 - 2C'xy + Bx^2 &= 0 \;\ldots\ldots\; (1),\\ Bz^2 - 2A'yz + Cy^2 &= 0 \;\ldots\ldots\; (2),\\ Cx^2 - 2B'zx + Az^2 &= 0 \;\ldots\ldots\; (3).\end{aligned}\right\}$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0 \;\ldots\ldots\; (4)$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0 \;\ldots\ldots\; (5),$$
$$B'y^2 + Bzx - C'zy - A'xy = 0 \;\ldots\ldots\; (6).$$

Between these (6), treated as simple equations, the six functions of x, y, z, viz. x^2, y^2, z^2, xy, xz, yz, treated as *independent* of each other, may be eliminated; the results may be seen, by mere inspection, to come out

$$ABC\,(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (*n. b.* not *irrelevant*) factor ABC, we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.$$

I may remark, that the equations (1), (2), (3), or (4), (5), (6), express the condition of

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy,$$

having a factor $\lambda x + \mu y + \nu z$; a general symbolical formula of which I am in possession for determining in general the condition of any polynomial of any degree having a factor, furnishes me at once with either of the two systems indifferently. The aversion I felt to reject *either*, led me to employ both, and thus was the occasion of the Dialytic Principle of Solution manifesting itself.

Ex. 2. $$Ax^2 + ayz + bzx + cxy = 0 \ldots\ldots\ldots (1),$$
$$My^2 + lyz + mzx + nxy = 0 \ldots\ldots\ldots (2),$$
$$Rz^2 + pyz + qzx + rxy = 0 \ldots\ldots\ldots (3).$$

Multiply equation (1) by $\beta y + \gamma z$, equations (2) and (3) by νz and κy respectively, and add the products together, we obtain terms of which y^2z and yz^2 are the only two into which x does not enter.

Make now the coefficients of each of these zero, and we have

$$a\gamma + l\nu + R\kappa = 0,$$
$$a\beta + M\nu + p\kappa = 0.$$

Let $\nu = a$, $\kappa = a$, then $\gamma = -(l + R)$, $\beta = -(M + p)$. Hence, multiplying as directed, and then dividing out by x, we obtain

$$(m\nu + b\gamma) z^2 + (r\kappa + c\beta) y^2 + (b\beta + c\gamma + n\nu + q\kappa) yz + A\beta xy + A\gamma xz = 0,$$

or by substitution,

$$\{ra - c(M + p)\} y^2 + \{ma - b(l + R)\} z^2 + \{an + aq - b(M + p) - c(l + R)\} yz - A(M + p) xy - A(M + p) xz = 0 \ldots\ldots\ldots (4).$$

Similarly, by preparing the equations so as to admit in turns of y and z as a divisor, we obtain

$$\{ma - l(R + b)\} z^2 + \{mr - n(A + q)\} x^2 + \{mc + mp - n(R + b) - l(A + y)\} xz - M(R + b) yz - A(A + q) xy = 0 \ldots\ldots\ldots (5),$$

$$\{rm - q(A + n)\} x^2 + \{ra - p(M + c)\} y^2 + \{rl + rb - p(A + n) - q(M + c)\} xy - R(A + n) xz - R(M + c) yz = 0 \ldots\ldots\ldots (6).$$

Between the six equations (1), (2), (3), (4), (5), (6), x^2, y^2, z^2, xy, xz, yz, may be eliminated; the result will be a function of nine letters {three out of each equation (1), (2), (3),} equated to zero. *Perhaps* the determinant may be found to contain a special factor of three letters; and if so, may be replaced by a simpler function of six letters only.

Ex. 3. To eliminate between the three general equations

$$Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy=0,$$
$$Lx^2+My^2+Nz^2+2Pyz+2Qzx+2Rxy=0,$$
$$fx+gy+hz=0.$$

By virtue of *one* of the two canons which limit the forms in which the letters can appear combined in the determinant of a general system of equations, we know that the determinant in this case (freed of irrelevant factors) ought to be made up in every term of eight letters (powers being counted as repetitions), viz. (A, B, C, D, E, F,) must enter in binary combinations (L, M, N, P, Q, R,) the same, whereas f, g, h, must enter in *quaternary* combinations.

To obtain the determinant, write

$$Ax^2+By^2+Cz^2+Dyz+Ezx+Fxy=0\ldots\ldots(1),$$
$$Lx^2+My^2+Nz^2+Pyz+Qzx+Rxy=0\ldots\ldots(2),$$
$$fx^2+gxy+hzx \qquad =0\ldots\ldots(3),$$
$$fxy+gy^2+hzx \qquad =0\ldots\ldots(4),$$
$$fxz+gyz+hz^2 \qquad =0\ldots\ldots(5).$$

We want one equation more of *three* letters between x^2, y^2, z^2, xy, xz, yz. To obtain this, write

$$(Ax+Ez+Fy)\,x_{,}+(By+Fx+Dz)\,y_{,}+(Cz+Dy+Ex)\,z_{,}=0,$$
$$(Lx+Qx+Ry)\,x_{,}+(My+Rx+Pz)\,y_{,}+(Nz+Py+Qx)\,z_{,}=0,$$
$$fx_{,}+ \qquad gy_{,}+ \qquad hz_{,}=0,$$

Forget that $x_{,}=x$, $y_{,}=y$, $z_{,}=z$, and eliminate x, y, z, we obtain

$$h\begin{Bmatrix}(Ax+Ez+Fy)\,(My+Rx+Pz)\\-(By+Fx+Dz)\,(Lx+Qz+Ry)\end{Bmatrix}$$
$$+g\begin{Bmatrix}(Cz+Dy+Ex)\,(Lx+Qz+Ry)\\-(Nz+Py+Qx)\,(Ax+Ez+Fy)\end{Bmatrix}$$
$$+f\begin{Bmatrix}(Nz+Py+Qx)\,(By+Fx+Dz)\\-(Cz+Dy+Ex)\,(My+Rx+Pz)\end{Bmatrix}=0.$$

This may be put under the form

$$\alpha x^2+\beta y^2+\gamma z^2+\alpha' yz+\beta' zx+\gamma' xy=0\ \ldots\ldots..(6),$$

where the coefficients are of the first order in respect to f, g, h, L, M, N, P, Q, R, A, B, C, D, E, F; in all of the third order.

Between the equations marked from (1) to (6), the process of linear elimination being gone through, we obtain as equated to zero a function of 5 + 3, or of eight letters, two belonging to the first equation, two to the second, and four to the third; so that the determinant is clear of all factorial irrelevancy.

Ex. 4. To eliminate x, y, z between the three equations

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0,$$
$$Lx^2+My^2+Nz^2+2L'yz+2M'zx+2N'xy=0,$$
$$Px^2+Qy^2+Rz^2+2P'yz+2Q'zx+2R'xy=0.$$

Call these three equations $U=0$, $V=0$, $W=0$, respectively. Write

$$x.U=0\ldots\ldots(1),\quad y.U=0\ldots\ldots(2),\quad z.U=0\ldots\ldots(3),$$
$$x.V=0\ldots\ldots(4),\quad y.V=0\ldots\ldots(5),\quad z.V=0\ldots\ldots(6),$$
$$x.W=0\ldots\ldots(7),\quad y.W=0\ldots\ldots(8),\quad z.W=0\ldots\ldots(9).$$

We have here nine unilateral equations: one more is wanted to enable us to eliminate *linearly* the ten quantities

$$x^3,\ y^3,\ z^3,\ x^2.y,\ x^2.z,\ x.y^2,\ x.z^2,\ x.y.z,\ y^2.z,\ y.z^2.$$

This tenth may be found by eliminating x, y, z, between the three equations

$$x(Ax+B'z+C'y)+y(By+C'x+A'z)+z(Cz+A'y+B'x)=0,$$
$$x(Lx+M'z+N'y)+y(My+N'x+L'z)+z(Nz+L'y+M'x)=0,$$
$$x(Px+Q'y+R'z)+y(Qy+R'x+P'z)+z(Rz+P'y+Q'x)=0;$$

for, by forgetting the relations between the bracketed and unbracketed letters, we obtain

$$(Ax+B'y+C'z)\left\{\begin{matrix}(My+N'x+L'z), & (Rz+P'y+Q'x),\\ -(Qy+R'x+P'z), & (Nz+L'y+M'x),\end{matrix}\right\}$$
$$+\ \&c.\ +\ \&c.\ =0,$$

which may be put under the form

$$\alpha x^3+\beta y^3+\gamma z^3+\delta x^2y+\ldots\ldots=0^{*}\ldots\ldots\ldots\ldots(10).$$

---

* We might dispense with a 10th equation, using the nine above given, to determine the ratios of the ten quantities involved to one another; and then by means of any such relations as

$$x^2y\times xy^3=x^2y^2\times x^2y^2,\ \text{or}\ x^3\times y^3=x^2y\times xy^2,\ \&c.$$

obtain a determinant. But it is easy to see that this would be made up of terms, each containing literal combinations of the 18th order.

Again, we might use five out of the nine equations to obtain a new equation free from y^3, y^2z, yz^2, z^3; *i. e.* containing x in every term: which being divided by x, and multiplied by y, or by z, would furnish a 10th equation no longer linearly involved in the 9 already found. The determinant, however, found in this way, would consist of 14-ary combinations of letters.

Finally, we might, instead of a system of ten equations, employ a system of 15, obtained by multiplying each of the given three by any 5 out of the 6 quantities x^2, y^2, z^2, xy, xz, yz; but the determinant, besides being not *totally* symmetrical, would contain combinations of the 15th order.

I may take this opportunity of just adverting to the fact, that the method in the text does in fact contain a solution of the equation

$$\lambda U+\mu V+\nu W=x^ry^sz^t,$$

where $r+s+t=4$, and λ, μ, ν are functions of the second degree in regard to x, y, z to be determined.

By eliminating linearly between the equations marked from (1) to (10), we obtain as zero a quantity of the twelfth order in all, being of the fourth order in respect to the coefficients of each of the three equations, which is therefore the determinant in its simplest form.

I have purposely, in this brief paper, avoided discussing any theoretical question. I may take some other opportunity of enlarging upon several points which have hitherto been little considered in the theory of elimination, such as the Canons of Form, —the Doctrine of Special Factors,—the Method of Multipliers as extended to a system of any order,—the Connexion between the method of Multipliers and the Dialytic Process,—the Idea of Derivations and of Prime Derivatives extended to ultra-binary Systems. For the present I conclude with the expression of my best wishes for the continued success of this valuable Journal.

22, *Doughty-Street, London, January* 30, 1841.

IX.—ON THE EXISTENCE OF POSSIBLE ASYMPTOTES TO IMPOSSIBLE BRANCHES OF CURVES.

Let $f(x, y) = 0$ represent the equation to any curve. Let $Ax^{\alpha} + Bx^{\beta}$ be the two first terms in a development for y by descending powers of x. Then if $\alpha=1$, $\beta=0$, and A, B, be both possible quantities, the equation

$$y = Ax + B$$

corresponds to a rectilinear asymptote, within the plane of x, y, to an infinite branch of the curve. In case all the coefficients of the descending powers of x, after the two first, be also possible, then the branch lies wholly within the co-ordinate plane from some point at a finite distance from the origin to its ultimate coalescence with the asymptote. If, however, any of the following coefficients be impossible, then the branch will be wholly without the plane of co-ordinates between such limits. We propose to illustrate the case of the impossible coefficients, by the discussion of the equations to two appropriate curves.

Suppose (λ) (μ) (ν) to be particular values of λ, μ, ν, which satisfy the proposed equation, the general values are of the form

$$(\lambda) + AV - BW,$$
$$(\mu) + CW - AU,$$
$$(\nu) + BU - CV,$$

U, V, W mean the same as above: A, B, C are arbitrary constants. (λ) (μ) (ν) may easily be found by *analysing* the method applied to example (4).

Ex. 1. Take the equation

$$x^2y^2-2bx^2y-b^2y^2-2a^2xy+b^2x^2+2b^3y+2a^2bx+2a^4-b^4=0 \ldots (1);$$

arranging by powers of y, we have

$$(x^2-b^2)\,y^2-2\,(bx^2+a^2x-b^3)\,y+b^2x^2+2a^2bx+2a^4-b^4=0,$$

and therefore, after the execution of obvious simplifications,

$$(x^2-b^2)^2\,y^2-2\,(x^2-b^2)\,(bx^2+a^2x-b^3)\,y+(bx^2+a^2x-b^3)^2$$
$$=2a^4b^2-a^4x^2,$$

and therefore

$$(x^2-b^2)\,y=bx^2+a^2x-b^3\pm a^2\,(2b^2-x^2)^{\frac{1}{2}} \ldots\ldots (2).$$

Suppose now that $x=\pm b$; then, since the coefficient of y becomes zero, and the right-hand member of the equation remains finite, it is clear that $y=\infty$. Hence the curve has two asymptotes, whose equations are

$$x=\pm b \ldots\ldots\ldots\ldots (3).$$

Again, suppose that x is indefinitely great; then retaining only the highest power of x in the coefficient of y and in the right-hand member of the equation, we have

$$x^2y=bx^2,$$

and therefore $\qquad y=b \ldots\ldots\ldots\ldots (4),$

which is the equation to an asymptote.

It is evident, however, from the equation (2), that y has impossible values for all indefinitely large values of x. Hence the equation (4) belongs to the possible asymptote of an impossible branch of the curve.

In order to place in the clearest point of view the character of the impossible branch, we will suppose x to be susceptible of every degree of quantitive magnitude affected by the sign $+$ or $-$, and transform accordingly the equation (1) from affectional to quantitive co-ordinates. The formulæ of transformation, (see a paper in the 9th number of this Journal, on the General Theory of the Interpretation of Equations in Algebraic Geometry,) are the following:—

$$\left.\begin{array}{lll} x=a, & y=\beta\,(\cos 2s\pi+-^{\frac{1}{2}}\sin 2s\pi), \\ x'=a, & y'=\beta\cos 2s\pi, & z'=\beta\sin 2s\pi, \end{array}\right\} \ldots\ldots (5).$$

Substituting the expressions for x, y, in the equation (1) arranged by powers of y, and equating separately to zero the possible and the impossible parts of the result, we have

$$(a^2-b^2)\,\beta^2\cos 4s\pi-2\,(ba^2+a^2a-b^3)\,\beta\cos 2s\pi$$
$$+b^2a^2+2a^2ba+2a^4-b^4=0 \ldots\ldots (6),$$

and $\quad (a^2-b^2)\,\beta^2\sin 4s\pi-2\,(ba^2+a^2a-b^3)\,\beta\sin 2s\pi=0,$

or $\quad \{(a^2-b^2)\,\beta\cos 2s\pi-(ba^2+a^2a-b^3)\}\sin 2s\pi=0 \ldots (7).$

The equation (7) is satisfied by either of the two relations

$$\sin 2s\pi=0 \ldots\ldots\ldots (8),$$

$$\text{or}\quad (a^2-b^2)\,\beta\cos 2s\pi-(ba^2+a^2a-b^3)=0\ldots\ldots(9).$$

From (8) we have $2s\pi=\lambda\pi$, where λ is any integer whatever; hence from (5) and (6) we get

$$x'=a,\quad y'=\beta\cos\lambda\pi,\quad z'=0,$$

$$(a^2-b^2)\,\beta^2\cos 2\lambda\pi-2\,(ba^2+a^2a-b^3)\,\beta\cos\lambda\pi$$
$$+b^2a^2+2a^2ba+2a^4-b^4=0,$$

and therefore

$$\left.\begin{array}{r}(x'^2-b^2)\,y'^2-2\,(bx'^2+a^2x'-b^3)\,y'+b^2x'^2+2a^2bx'+2a^4-b^4=0\\ \ldots\ldots\ldots(10),\\ \text{and}\quad z'=0\,;\end{array}\right\}$$

again, from (5), (6), (9), we have

$$(x'^2-b^2)\,y'=bx'^2+a^2x'-b^3\ldots\ldots(11),$$

and

$$(x'^2-b^2)\,(y'^2-z'^2)-2\,(bx'^2+a^2x'-b^3)\,y'$$
$$+b^2x'^2+2a^2bx'+2a^4-b^4=0,$$

and therefore, by combining the two,

$$(x'^2-b^2)\,(y'^2+z'^2)=b^2x'^2+2a^2bx'+2a^4-b^4\,;$$

whence by (11)

$$(x'^2-b^2)^2z'^2+(bx'^2+a^2x'-b^3)^2=(x'^2-b^2)(b^2x'^2+2a^2bx'+2a^4-b^4),$$

and therefore

$$(x'^2-b^2)^2\,z'^2=a^4\,(x'^2-2b^2)\ldots\ldots\ldots(12).$$

The equations (10) represent the branches SKs, S′K′s', in the plane of x', y', (See Fig. 3.) The straight lines SS′, ss', which belong to the equations (3), are asymptotes to these branches.

The equations (11), (12), represent the branches TLKl, T′L′K′l', of which KIT, K′I′T′, are the projections on the plane of x', y'. The straight line TT′ corresponding to the equation (4), is an asymptote to both these branches.

The equation (1) may be seen in Cramer's *Analyse des Lignes Courbes*, where, having defined the locus of (4) as being an equation of the form which corresponds to a rectilinear asymptote, he observes that, since the values of y in (1) for indefinitely great values of x, are impossible, "cette prétendue asymptote TT′ n' est accompagnée d' aucune branche infinie de la courbe."

Ex. 2. Take as another instance the equation

$$x^4\,(y-bx-c)^2=a^2-x^2\ldots\ldots\ldots(1).$$

Writing this equation under the form

$$(y-bx-c)^2=\frac{a^2-x^2}{x^4}\ldots\ldots\ldots(2),$$

it is clear that when x becomes indefinitely great

$$y=bx+c\ \ldots\ldots\ldots\ldots\ldots(3),$$

which is therefore the equation to an asymptote in the primary co-ordinate plane. But it is clear that when x is very great the values of y in (2) are impossible. Hence the equation (3) belongs to the possible rectilinear asymptote of an impossible branch.

If, as in the former example, we transform the equation (1), in which x is supposed to receive every degree of quantitive magnitude affected by $+$ or $-$, from affectional to quantitive co-ordinates, we shall obtain the two following pairs of equations:

$$\left.\begin{aligned} x'^4(y'-bx'-c)^2 &= a^2-x'^2, \\ z' &= 0, \end{aligned}\right\} \ldots\ldots\ldots (4),$$

$$\text{and}\quad \left.\begin{aligned} y' &= bx'+c, \\ x'^4z'^2 &= x'^2-a^2, \end{aligned}\right\} \ldots\ldots\ldots (5).$$

The equations (4) correspond to the branches BAb, B′A′b', (Fig. 4,) in the plane of x', y', to which the axis of y' is asymptotic; and the equations (5) to the branches TKAk, T′K′A′k', which have an asymptote TT′ in the plane of x', y', their projections upon this plane coinciding with TT′.

W. W.

X.—MATHEMATICAL NOTES.

1. A short mode of reducing the square root of a number to a continued fraction.

The common mode of proceeding by successive similar operations may be thus reduced to a rule, which will best be described by an instance. Let the number be 43,

$$\sqrt{43} = 6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5} + \text{\&c.}}}}}$$

$$\begin{array}{c|cccccccccc|l} 6 & 1 & 5 & 4 & 5 & 5 & 4 & 5 & 1 & 6 & 6 & 1, \text{ \&c.} \\ 1 & 7 & 6 & 3 & 9 & 2 & 9 & 3 & 6 & 7 & 1 & 7, \text{ \&c.} \\ 6 & 1 & 1 & 3 & 1 & 5 & 1 & 3 & 1 & 1 & 12 & 1, \text{ \&c.} \end{array}$$

Set down always in the first column the integers of the root, a unit, and the integers again. Form each column after the first from the preceding in the following manner:—

$$c, \quad c', \quad a' = \frac{43 - c^2}{a}, \text{ which is always integer,}$$

$$a, \quad a', \quad b' = \text{integer of } \frac{6+c}{a'},$$

$$b, \quad b', \quad c' = a'b' - c.$$

The first figure of the third row is always the integer, and the last figure twice the integer; and the intermediate figures always shew the same series, whether reckoned from the beginning or the end. As soon as a succession of two similar numbers is seen in the first or second rows, it is a sign that the middle of the period is attained: if the reiteration take place in the first row, there will be an odd number of figures the same in the middle of the period; and if in the second row, an even number. In the last column but one of the whole period, $6+c$ will be divisible by a', but the converse is not true.

A. D. M.

2. *Irrationality of* $\varepsilon = 2{\cdot}71828\ldots$ To the demonstration usually given, that ε is incommensurable, may be added this—that it cannot be the root of a quadratic equation, the coefficients of which are rational. If so it would satisfy the equation

$$a\varepsilon + \frac{b}{\varepsilon} = c,$$

a being a positive integer, and b and c integers either positive or negative. If then we replace ε by its equivalent series, and then multiply both sides by $1.2\ldots n$, we find

$$\frac{a}{n+1}\left(1 + \frac{1}{n+2} + \&c\right) \pm \frac{b}{n+1}\left(1 - \frac{1}{n+2} + \&c.\right) = \mu$$

μ being an integer. But we can always make the factor $\pm \dfrac{b}{n+1}$ positive by taking n even when b is negative, and *vice versâ*. If now we suppose n very large, the first side of the equation is a fraction which cannot be equal to the integer μ. Hence follows the proposition.—*Liouville's Journal, May* 1840.

ERRATUM.

Page 232, line 25 from the bottom, *for* Dec. 1840, *read* Feb. 1840

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I.—ON FRINGES OF INTERFERENCE PRODUCED BY OBLIQUE REFLEXION AT THE SURFACE OF A SMALL MIRROR.

By ANDREW BELL, *Edinburgh.*

THE object of this paper is to determine the circumstances connected with the production of fringes of interference by oblique reflexion at a plane surface, whose projection on a plane perpendicular to the direction of the light, is narrow.

These fringes are produced by light diverging from a minute origin, as a luminous point or line, and are the same as those produced by transmission through an oblique rectangular aperture of the same breadth and inclination to the incident light as the mirror.

The fringes are similar to those produced by narrow perpendicular apertures, with this peculiarity, that the fringes which are nearer to the plane of the inclined aperture or mirror are broader than those which are more distant; the intervals forming an increasing series from one side to the other; the difference in the magnitude of the intervals being very apparent when the inclination or breadth of the mirror, or both, are very small.*

The reflected light is considered to consist of secondary waves emitted from every point of the mirror; and although the waves emitted from the surface at any instant are not then in the same phase, yet the portions of the front of the same primary wave that generate in succession a system of secondary waves, were in the

* Fresnel, in his memoir in the *Memoirs of the Institute* for 1821, alludes in general terms to the cases of oblique reflexion and transmission through oblique apertures; but he does not investigate the case. and he takes no notice of the varying magnitude of the intervals.

same phase when the front of the wave was in some previous position, or when it was emitted from the origin of light; and the paths can be reckoned from that position, or the origin.

If the fringes formed by a single straight edge are called a *single* system, and those formed by a narrow aperture, a *double* system; then the reflected partial waves are under all the necessary conditions for forming a double system of fringes, if the screen is at a sufficient distance from the mirror to allow the partial waves from the whole surface to interfere; and if at the same time the difference of their paths is not too great, reckoning them as formerly stated.

The disturbing effect at any point on a screen produced by the partial waves reflected from an element of the surface, can be estimated in the following manner:—The function $\sin \frac{2\pi}{\lambda}(vt - s)$, which, with a proper coefficient, expresses the state of the particles of ether at any instant along the course of a primary incident wave to some point on the mirror, will also, with its coefficient properly modified, express at the same instant the state of the particles along the course of a secondary wave, generated by the former at that point of the surface. If x is reckoned from the origin, the intensity of vibration at the point on the mirror is inversely as x; and if y is the distance of the point on the screen from that on the mirror, the intensity of vibration at the former point is inversely as y, when that at the latter point is constant: therefore the intensity of vibration at the point on the screen will be inversely as xy, which is nearly constant.

Hence, if Δw denote the breadth of an element of the mirror, and if $s = x + y$; also, if D be the disturbance at the point on the screen, then

$$D = C\,\Sigma \sin \frac{2\pi}{\lambda}(vt - s)\,\Delta w.$$

To determine the intervals on a screen, the section of which by the plane of incidence is a circle, its centre being in the middle of the mirror.

Let RM (fig. 1) be the mirror, O the luminous origin, FQF′ the section of the screen, m the middle of the mirror, mQ the direction in which a wave is reflected which is incident in the direction Om; also, let Ow, wF, be the path of any other wave, and let

$$\begin{aligned} RM &= 2b, & wF &= u,\\ mw &= w, & mF &= u',\\ Ow &= r, & \text{angle } OmC &= \beta,\\ Om &= r', & \text{and } FmQ &= \theta. \end{aligned}$$

In the triangles Omw, Fmw,

$$r^2 = r'^2 + w^2 + 2r'w \cos \beta,$$
$$u^2 = u'^2 + w^2 - 2u'w \cos(\beta - \theta).$$

Hence $$r=r'+w\cos\beta+\frac{\sin^2\beta}{2r'}w^2+\ldots$$

and $$u=u'-w\cos(\beta-\theta)+\frac{\sin^2(\beta-\theta)}{2u'}w^2+\ldots$$

But when a double system of fringes is produced, the greatest value of $w\sin\beta$ must not exceed $\frac{1}{10}$ of an inch if the screen is not more than a few feet distant, suppose not less than 50 inches;* hence, since θ is always small, the third terms in the values of r and u will at most be only a few multiples of λ, and may therefore be omitted. Hence

$$r+u=r'+u'+\{\cos\beta-\cos(\beta-\theta)\}\,w.$$

By the usual method, the intensity of light at any point on the screen is found to be

$$\frac{C^2\sin^2\frac{2\pi}{\lambda}\{\cos\beta-\cos(\beta-\theta)\}\,b}{\frac{4\pi^2}{\lambda^2}\{\cos\beta-\cos(\beta-\theta)\}^2},$$

therefore the points on the screen at which the dark bands appear are determined by the equation

$$\frac{2\pi}{\lambda}\{\cos\beta-\cos(\beta-\theta)\}\,b=n\pi,$$

whence $$\cos(\beta-\theta)=\cos\beta-\frac{n\lambda}{2b}.$$

When $n=0$, $\cos(\beta-\theta)=\cos\beta$; hence $\theta=0$. From this result it would appear that there is a dark band at Q, whereas it is known by observation that the middle point is bright. The ratio of the intensity at this part, and at the bright bands, is determined afterwards.

When n is *negative*, $\cos(\beta-\theta)>\cos\beta$; hence θ is in this case *positive*, and the fringes lie on the side of Q, that is, next to the plane of the mirror towards F.

When n is *positive*, $\cos(\beta-\theta)<\cos\beta$; hence θ is then *negative*, and the fringes lie on the other side of Q towards F′.

By substituting successively for n the values .. 3, 2, 1, 0, −1, −2, −3,... the cosine of $(\beta-\theta)$ has a series of increasing values in equidifferent progression, the common difference being $\frac{\lambda}{2b}$. But if $d.\cos(\beta-\theta)=\frac{\lambda}{2b}$, it is constant, and hence $d\theta\propto\frac{1}{\sin(\beta-\theta)}$;

* Fresnel observed the double system of fringes formed by perpendicular apertures of 1 and even of 1½ centimetres in breadth; but the lens by which he observed them was held at a great distance from the aperture, and at such distances the interference is very imperfect.

and since in the above series $\cos(\beta-\theta)$ is increasing, $(\beta-\theta)$ is diminishing, and hence also $\sin(\beta-\theta)$; therefore the values of $d\theta$, or the intervals on the screen, form an increasing series. This result is confirmed by observation.

When the screen is perpendicular to the plane of the mirror, it is evident that the intervals on it are nearly proportional to

$$d.\tan(\beta-\theta)=-\frac{d.\cos(\beta-\theta)}{\sin(\beta-\theta)\cos^2(\beta-\theta)};$$

that is, the intervals are inversely as

$$\sin(\beta-\theta).\cos^2(\beta-\theta).$$

Let p and q be given values of the sine and cosine of $(\beta-\theta)$, and let q receive the small increment nk, $\left(k \text{ being } =\frac{\lambda}{2b}\right)$; then, neglecting terms containing k^2, q^2 becomes q^2+2nqk, and

$$p=\surd(1-q^2)-\frac{nqk}{\surd(1-q^2)}=p-mk,$$

if $m=\frac{nq}{p}$. Hence for small values of n, as 1, 2, 3, or 4, the increment of pq^2 is, unless in extreme cases, nearly constant; and hence the successive values of

$$\sin(\beta-\theta).\cos^2(\beta-\theta)$$

for successive values of n, are nearly in equidifferent progression; and their reciprocals, for a few terms, are therefore also nearly in equidifferent progression. And therefore the intervals on the perpendicular screen form a similar progression.

The quantity $\sin(\beta-\theta).\cos^2(\beta-\theta)$ increases with $(\beta-\theta)$ till it attains its maximum, when $\sin^2(\beta-\theta)=\frac{1}{3}$, and it then diminishes; hence the intervals on the plane screen for greater values of $(\beta-\theta)$ vary in a contrary order, the greater intervals being then farther distant from the plane of the mirror.

The positions of the bright bands are found from the equation

$$\frac{2\pi}{\lambda}\{\cos\beta-\cos(\beta-\theta)\}\,b=\tfrac{1}{2}n\pi,$$

n being an odd number: whence

$$\cos(\beta-\theta)=\cos\beta-\frac{n\lambda}{4b},$$

and the successive increments of $\cos(\beta-\theta)$ are constant for the successive odd values of n, and are $=\frac{\lambda}{2b}$, as for the dark bands.

The expression for the intensity of the illumination at the middle point, for which $\theta=0$, assumes the form $\frac{0}{0}$; its proper value on the

principle of vanishing fractions will be found to be C^2b^2. Also the values of the same quantity for the bright bands in order, are

$$\left(\frac{2Cb}{\pi}\right)^2, \quad \left(\frac{2Cb}{3\pi}\right)^2, \quad \left(\frac{2Cb}{5\pi}\right)^2, \text{ \&c.}$$

$$\text{for} \quad n = \pm 1, \quad \pm 2, \quad \pm 3, \ldots\ldots$$

Therefore the intensities at the middle point, and at the first, second, third, ... bright bands are proportional to

$$\frac{1}{4}, \quad \frac{1}{\pi^2}, \quad \frac{1}{(3\pi)^2}, \quad \frac{1}{(5\pi)^2}, \ldots\ldots$$

the intensities, therefore, diminish rapidly, and soon become insensible.

In a note at p. 404 of his *Memoir on Diffraction*, Fresnel proves that the intensities at two different points produced by the secondary waves, proceeding from the front of the same primary wave, are inversely as the distances of the points from the origin of light; and therefore these intensities are at least proportional to those produced by the primary wave itself when it reaches these points. These results are obtained by supposing the front of the wave to be divided in a particular manner; and by a similar method it can be proved, that the intensities at a point produced by the partial wave from the fronts of two primary waves from the same origin are equal.

Let Wr, W$'r'$, (fig. 2.) be the fronts of two waves proceeding from O, and P any point. Let Wm, W$'m'$, be two arcs of circles, of which P is the centre, and Wn, W$'n'$, two tangents. Also let Pr exceed PW as much as Pr' exceeds PW$'$; let the distance of W from O and P be x and y, W$n = z$, $mr = s$; and let x', y', z', and s' represent the similar lines for the wave W$'r'$; then as the arcs Wr, W$'r'$, are very small,

$$mn = \frac{z^2}{2y} \text{ nearly}, \; nr = \frac{z^2}{2x}, \text{ hence } s = \frac{(x+y)z^2}{2xy};$$

$$\text{therefore } z^2 = \frac{2xy}{x+y}\, s, \text{ and hence } z'^2 = \frac{2x'y'}{x'+y'}\, s,$$

$$\text{for } mr = m'r'; \text{ but } x + y = x' + y':$$

$$\text{therefore } z^2 : z'^2 = xy : x'y'.$$

But the intensities of vibration at W and W$'$ are inversely as x and x'; and the intensities produced at P by partial waves from equal portions of the fronts of the two waves, are inversely as y and y', were the intensities at W and W$'$ equal; also the two circular portions of the fronts, of which z and z' are the radii, are, as shown above, as xy and $x'y'$; and the waves proceeding from these portions are in exactly similar states in regard to accordance or discordance, if the fronts are in the same phase; therefore the

intensities of vibration at P, caused by partial waves from these spaces, being denoted by v and v', it follows that

$$v : v' = \frac{z^2}{xy} : \frac{z'^2}{x'y'} = 1 : 1 ;$$

that is, they are equal.

Pairs of lines, like Pr, Pr', may be drawn from P to the fronts of the waves, such that their differences from the corresponding normals PW, PW', shall be equal; and the same proportions can be proved to exist in reference to them. And since very small portions of the fronts of the two waves are considered, and xy, $x'y'$, are constant, the corresponding zones into which the fronts would be divided would be proportional; and the waves emitted from the corresponding zones would be in analogous states; and hence the intensities produced by them at P would also be equal. Therefore, the intensities produced at P by partial waves from the fronts of the two primary waves are equal.

In these calculations it is supposed, that the extent of the portion of the front of a wave from which partial waves of sensible intensity proceed, is very small, and that the intensities are uniform, which agrees with observation. But it also implies that the limit of this small surface is determined by a certain difference in the lengths of the paths of the extreme partial waves. It is known by observation, that such a difference alone is a limit to perfect interference, as well as a certain very small inclination. It is also assumed, that at a given distance from the front of a wave, the intensity of vibration of a partial wave is proportional to that in the front of the wave, which is very probable.

The case of transmission through inclined rectangular apertures, is calculated in exactly the same manner as that of oblique reflexion.

If the image of the luminous origin be considered to be the new origin for direct light, and if the inclined mirror be considered to be an oblique aperture, and if partial waves be supposed to proceed from every point in the surface of the mirror, the calculations for this case will be identical with the preceding for oblique reflexion.

The supposition of the emission of secondary waves from the plane of the aperture is equivalent to this hypothesis:—In determining the effect of transmission through narrow oblique apertures, instead of the partial waves being supposed to proceed from the front of a primary wave, they may be conceived to be emitted from the consecutive line of common section of the front with the plane of the narrow aperture, as it advances through the aperture; and the intensity at any point produced by the transmitted light may be calculated by estimating the effect produced by these secondary waves.

I have calculated the breadth of one of the fringes on a circular screen, corresponding to the data of one of my former experiments,

and have found the agreement to be within the limits of what I would consider the unavoidable errors of observation, at least by the means that I have of observing. The difference was about $\frac{1}{200}$ of an inch, or about the tenth part of an interval. It would, I believe, with more perfect apparatus than I have, be possible to determine the intervals more accurately; although there is a limit to the necessary degree of accuracy of apparatus, from the fact, that the eye alone can determine the position of the lines of minimum intensity; and this cannot be done with very great precision.

Most of the experiments that I performed, were only for the purpose of satisfying myself in regard to the general appearance of the fringes. I believe I took accurate measurements only in two or three instances, for the purpose of comparing the observed and calculated intervals

The manner in which I performed the experiments was this:— P (fig. 3.) is a candle or a small mirror reflecting the sun's light. The light passes through O, a vertical aperture in a sheet of pasteboard, and its breadth is about $\frac{1}{40}$ of an inch. M'Q is a screen, a sheet of white pasteboard, placed vertically, and perpendicular to the plane MD of the mirror. When the sun's light is used, the fringes are seen on both sides of Q on the pasteboard; when the light of a candle is used, a lens of about 4 inches focal length, and 3 inches in diameter, is placed at Q; and when the eye is situated behind it, nearly in the principal focus, the fringes are seen very distinctly, and can be measured by applying a pair of finely pointed compasses to the anterior surface of the lens. The measurement by this means will not be so exact as by a micrometer; but the error of observation may not much exceed that arising from estimating by the eye the points of minimum intensity. The breadth of the reflector was measured also by compasses, which were applied to an accurate scale, and by means of a lens the measures could be easily estimated to the $\frac{1}{200}$ part of an inch.

The inclination β can be found with considerable accuracy by taking the lines mm', mQ, equal, and about 100 inches; and measuring also the line m'Q, and then calculating β.

If RM is the breadth of an inclined aperture, and if the point A in the line OO′, perpendicular to its plane, be found where the extension of its plane meets OO′; then by placing the mirror RM so that the image of O observed from Q may coincide with O′, AO′ being = AO, the mirror will then have the same inclination; and the breadth of the fringes produced by them will be the same. I have at least convinced myself by some experiments that this is the case, though I have not got measurements taken with sufficient precision for publishing them.

One circumstance is apt to mislead if not attended to. If the substance in which the oblique aperture is made is of sensible thickness, and not thinned off to a fine edge, then, the light proceeding in the direction at R, (see fig. 4.) the breadth of the aperture is evidently MR instead of NR; when pasteboard is used, this circumstance produces a very sensible effect with small inclinations.

The mirrors that I used were of plate glass, covered on one side with black wax, and one of speculum metal, $\frac{6}{10}$ of an inch broad.

The formula for the intervals shows that they are independent of the distance of the mirror or aperture from the light, as is the case with narrow obstacles and perpendicular apertures.

II.—NEW SOLUTION OF A CUBIC EQUATION.

By J. Cockle.

There is a soluble form of the perfect cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

to which all cubic equations may be reduced. For if $3ac = b^2$, and the equation be put under the form

$$-x^3 = ax^2 + bx + c,$$

and each side be multiplied by $3ab$,

$$-3abx^3 = 3b.a^2x^2 + 3b^2.ax + b^3, \text{ (since } 3abc = b^3\text{)};$$

adding a^3x^3 to each side, and extracting the cube root,

$$x\sqrt[3]{a^3 - 3ab} = ax + b;$$

therefore

$$x = \frac{b}{\sqrt[3]{a^3 - 3ab} - a} \text{ ....(1).}$$

In *any* cubic where $3ac$ is not equal to b^2, put $y + z$ for x, and

$$x^3 + ax^2 + bx + c = 0 \text{ becomes}$$

$$y^3 + (3z + a)y^2 + (3z^2 + 2az + b)y + z^3 + az^2 + bz + c = 0,$$

$$\text{or } \quad y^3 + Ay^2 + By + C = 0,$$

and it is required that $\quad 3AC = B^2.$

Putting for A, C, and B, their values in terms of z, &c., and equating; the powers of z above the second vanish, and we have

$$(a^2 - 3b)z^2 + (ab - 9c)z + (b^2 - 3ac) = 0,$$

$$\text{or} \quad z^2 + \frac{ab - 9c}{a^2 - 3b} z + \frac{b^2 - 3ac}{a^2 - 3b} = 0;$$

therefore z is known, and by (1)

$$y = \frac{B}{\sqrt[3]{A^3 - 3AB} - A},$$

$$x = y + z = \frac{B}{\sqrt[3]{A^3 - 3AB} - A} + z.$$

Now $A^3 - 3AB = A(A^2 - 3B)$

$$= (3z + a)\{(3z + a)^2 - 3(3z^2 + 2az + b)\}$$

$$= (3z + a)(a^2 - 3b);$$

therefore

$$y = \frac{3z^2 + 2az + b}{\sqrt[3]{(a^2 - 3b)(3z + a)} - (3z + a)} = \frac{(3z + a)z + az + b}{\sqrt[3]{(a^2 - 3b)(3z + a)} - (3z + a)}.$$

Hence, if $3z + a = \alpha$,

$$x = \frac{\alpha z + az + b}{\sqrt[3]{(a^2 - 3b)\,\alpha} - \alpha} + z \;\ldots\ldots\ldots\; (2),$$

and (1) may be put under the form

$$x = \frac{b}{\sqrt[3]{(a^2 - 3b)\,a} - a}.$$

As an example, let $x^3 - 4x^2 + 6x - 3 = 0$.

Here $3ac = 36 = 6^2 = b^2$,

$$\text{therefore } x = \frac{b}{\sqrt[3]{a^3 - 3ab} - a} = \frac{6}{\sqrt[3]{8} + 4} = \frac{6}{6} = 1.$$

Again, $x^3 - 7x^2 + 17x - 14 = 0$.

Here $3ac$ is not equal to b^2; therefore we must apply the second method. The equation for finding z is -

$$z^2 + \frac{ab - 9c}{a^2 - 3b} z + \frac{b^2 - 3ac}{a^2 - 3b} = 0,$$

$$\text{or} \quad z^2 + \frac{-119 + 126}{49 - 51} z + \frac{289 - 294}{49 - 51} = 0,$$

$$\text{or} \quad z^2 - \frac{7}{2} z = -\frac{5}{2}, \text{ therefore } z = 1 \text{ or } \frac{10}{4}.$$

Take $z = 1$, $\alpha = 3z + a = -4$,

and putting $a^2 - 3b = n$,

$$x = \frac{(\alpha + a)z + b}{\sqrt[3]{n\alpha} - \alpha} + z = \frac{-11 + 17}{2 + 4} + 1 = \frac{6}{6} + 1 = 2;$$

therefore $x = 2$.

$3\frac{b}{a^2}$ might have been used as a multiplier in the commencement instead of $3ab$.

KKK

III.—ON THE FORM OF A BENT SPRING.*

In the following papers it is proposed to discover the differential equation to the curve formed by a bent spring, upon the supposition that the particles of the spring are compressible as well as extensible; the present assumed law of tension being only true when the particles can be extended but not compressed.

A spring is supposed to be made up of an indefinite number of indefinitely thin laminæ; and the laminæ composed of an indefinite number of indefinitely short elementary particles, as in fig. 5.

Axioms which are founded upon experiment.

1. If a right line rr', passing through the middle points o, o' of two or more contiguous elementary particles p_1p', qp_1', be perpendicular to their forces pp', p_1p_1', before tension, it shall be so during tension. Did this law not hold the laminæ of the spring would have a sliding motion, and the form of the bent spring could not be made the subject of mathematical investigation.

Cor. All the laminæ of a bent spring have the same normals at the same distances from the ends of the spring; and likewise the centre of curvature of one is the centre of curvature of all the contiguous elementary particles.

2. The particles of springs are extensible and compressible in direction of the laminæ, and it is owing to this state of extension and compression only, that the spring is elastic.

3. If δs be the length of an elementary particle of a lamina before tension, $\delta s'$ after tension; and if T be the tension, and e the elastic modulus,

$$\frac{\delta s' - \delta s}{\delta s} = \frac{\mathrm{T}}{e}.$$

To find the tension of a bent spring at any point, and the force with which it tends to resist bending, let AC (fig. 6.) be the bent spring, QO, Q'O, normals through the middle points of two adjacent laminæ: then these bisect all the contiguous elementary laminæ. Let ll', $l'l''$, be the halves of those adjoining elementary laminæ which are neither extended nor compressed, since the result of bending the spring in the ordinary way is to compress those laminæ towards the concave part, and extend those towards the convex.

Then, if $l'rr'$ be parallel to QO, $l'r''r'''$ parallel to Q'O, these lines define the original boundaries of the elementary particles; consequently if x be the distance of the extremity of a particle from

* From a Correspondent.

ll', that particle has been extended $\frac{xds}{r}$, if $ds = ll'$, $r =$ rad. of curvature; therefore the force of tension arising from that extension tending to close the extended laminæ, is $\frac{cx\,\delta x\,.\,ds}{rds}$, if δx be its thickness. Also, its moment tending to turn Qrl' about l into its former position, is $\frac{cx^2\,\delta x}{r}$; therefore $\frac{ca^2}{2r}$ is the force tending to join the spring, if $l'r = a$; and $c\left(\frac{t-a}{2r}\right)^2$ is the force tending to separate the spring if $r'r = t$; therefore $c\left(\frac{a^2-(t-a)^2}{2r}\right)$ is the total force tending to join the spring, or acting upon Qr''' in direction ll'.

Also, $c\left(\frac{a^3+(t-a)^3}{3r}\right)$ is the total force by which Qr' would turn about l' into its former position, were it not that another force keeps it stretched. If the spring be incompressible, this latter force $= \frac{ct^3}{3r}$, which agrees with the ordinary received force.

To find the differential equation to the curve, the thickness of the spring being inconsiderable.

Let P (fig. 7.) be a neutral point, supposed on the spring, R the force applied at the end of the spring; X, Y, the resolved parts of R, in direction of axes of x and y; x, y, the co-ordinates of P: therefore we have

$$X\frac{dx}{ds} + Y\frac{dy}{ds} = -c\left\{\frac{a^2-(t-a)^2}{2r}\right\} \;\ldots\ldots\; (1),$$

$$Xy + Yx = c\left\{\frac{a^3+(t-a)^3}{3r}\right\} \;\ldots\ldots\ldots\; (2).$$

In most cases, since a is very small, c must be very large, and the neutral point will lie nearly in the middle of the spring; in this case the ordinary equations hold.

Between (1) and (2) we must eliminate a, and the resulting equation is the equation required. It is to be observed that the resulting equation is the equation to the neutral line. The equation is of the second order, since

$$r = -\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

It is generally more convenient to make the direction of R the axis of y.

In this case we have

$$\frac{\mathrm{Y}\,dy}{ds} = -\,c\left(\frac{a^2-(t-a)^2}{2r}\right) = \left(\frac{t^2-2at}{2r}\right).c,$$

$$\mathrm{Y}x = \quad c\left(\frac{a^3+(t-a)^3}{3r}\right) = c\left(\frac{t^3-3at^2+3a t^2}{3r}\right).$$

Cor. If it be required to discover the law of thickness of a spring, that when bent by a given force it may assume a given form, let $y=f(x)$ be the equation to the curve. We must substitute for $\frac{dy}{ds}$ its value in terms of x, and for r its value in terms of x, whence eliminating a we shall obtain $t_1 = \phi(x)$.

IV.—SINGULAR POINTS IN SURFACES.

By D. F. Gregory, M.A. Fellow of Trinity College.

The nature of those points in surfaces which are analogous to multiple points in curves, has not, so far as I know, been hitherto discussed in any work on Analytical Geometry. Leroy, in his *Analyse Appliquée*, and in his *Géometrie Descriptive*, does little more than indicate their existence; and Monge and Dupin, although they have given much attention to those singular points called *umbilici*, have not spoken at all of the points which are the object of the present article. The subject is, however, one of some interest, from the light which it throws on the form of certain surfaces; and in physical investigations Sir W. Hamilton has shown by his researches on the form of the Wave Surface, that such points are of considerable importance.

In the first place, it is necessary to define precisely what is meant by a *singular* point in the sense in which I use that phrase. If

$$\mathrm{F}(x, y, z) = 0 \ldots\ldots \ldots(1),$$

be the equation to a surface, then those points for which all the differential coefficients of F, with respect to x, y, and z, below a certain order, vanish, I call singular points; and the values of x, y, z, which satisfy these conditions, I shall call the *critical* values. We shall have to consider chiefly points for which the differential coefficients, of the first order only, vanish, so that the characteristic of such points is, that the equations

$$\frac{d\mathrm{F}}{dx}=0, \quad \frac{d\mathrm{F}}{dy}=0, \quad \frac{d\mathrm{F}}{dz} = 0 \ldots \ldots \ldots(2),$$

are satisfied by simultaneous values of x, y, z, which also satisfy the equation to the surface (1). When the conditions (2) do not hold, we know that the locus of the tangent lines to the surface at a given

point is a plane, which is called the tangent plane. When the conditions (2) do hold, the direction cosines of the tangent take the form $\frac{0}{0}$, and are therefore indeterminate, showing that there may be more than one tangent plane at the point: and we shall see that there are in general an infinite number of such planes, forming by their intersections a tangent *cone*.

Let us now investigate the locus of the tangent lines at a singular point. This we shall do by the same method as that employed in Vol. I. p. 135; and for the convenience of our future operations, let us put

$$U = \frac{dF}{dx}, \qquad V = \frac{dF}{dy}, \qquad W = \frac{dF}{dz},$$

$$u = \frac{d^2F}{dx^2}, \qquad v = \frac{d^2F}{dy^2}, \qquad w = \frac{d^2F}{dz^2},$$

$$u' = \frac{d^2F}{dy\,dz}, \qquad v' = \frac{d^2F}{dx\,dz}, \qquad w' = \frac{d^2F}{dx\,dy}.$$

Let then the equation to a line passing through the point x, y, z, in the surface, be

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} = r \ldots\ldots\ldots (a),$$

x', y', z', being the current co-ordinates of the line. This line will generally be cut by the surface in one or more other points. Let the co-ordinates of the nearest of these be x_1, y_1, z_1, then from equation (a) we have

$$x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr.$$

Substituting these values in the equation to the surface, it becomes

$$F\{x + lr, \quad y + mr, \quad z + nr\} = 0.$$

Expanding this, considering lr, mr, nr, as the increments of x, y, z, we have

$$0 = F(x, y, z,) + r\,(lU + mV + nW) +$$
$$+ \frac{r^2}{1.2}\{l^2u + m^2v + n^2w + 2(mnu' + lnv' + lmw')\} + \frac{r^3}{1.2.3}\{\quad\}.$$

But from the equation to the surface

$$F(x, y, z,) = 0;$$

and when the point is a singular point,

$$U = 0, \quad V = 0, \quad W = 0;$$

therefore the equation is reduced to

$$0 = \frac{1}{1.2}\{l^2(u) + m^2(v) + n^2(w) + 2(mn(u') - ln(v') + lm(w')\}$$
$$+ \frac{r}{1.2.3}\{\quad\}.$$

where the differentials are inclosed in parentheses to indicate the

values they receive when the critical values of x, y, z, are substituted in them. When the line becomes a tangent, the points x, y, z, x_1, y_1, z_1, ultimately coincide, and r becomes indefinitely small; in which case the preceding equation becomes ultimately

$$l^2(u) + m^2(v) + n^2(w) + 2\{mn(u') + ln(v') + lm(w')\} = 0;$$

whence eliminating l, m, n, by means of equation (a), we find

$$(u)(x'-x)^2 + (v)(y'-y)^2 + (w)(z'-z)^2 + 2\{(w')(y'-y)(z'-z) + (v')(x'-x)(z'-z) + (w')(y'-y)(x'-x)\} = 0 \ldots\ldots(3),$$

as the equation to the locus of the tangent lines. This is evidently in general a cone of the second degree, though with certain values of the coefficients, it may represent two planes or a straight line. On transferring the origin to the singular point, the equation (3) may be put under the form

$$(u)x^2 + (v)y^2 + (w)z^2 + 2\{(u')yz + (v')xz + (w')xy\} = 0 \ldots\ldots(4).$$

If we suppose that all the second differential coefficients also vanish for the critical values of the variables, we should easily find that the locus of the tangent lines at a singular point is a cone of the third order, and so on in succession to any order. The forms of the equations are easily found by taking the corresponding differential of F(x, y, z), and substituting in it x, y, z, for dx, dy, dz. It is unnecessary to write down these expressions, which are rather long, as I shall scarcely have occasion to use them. Before proceeding to give some applications of the formula which has just been investigated, I shall make one or two remarks on the difference, with respect to singular points, between plane curves and surfaces. In the first place it will be readily seen that the three equations (2) may be satisfied by some relation between the variables, without assigning any particular value to each. If we suppose the relation to be given by the equation

$$\phi\,(x, y, z,) = 0,$$

the intersection of the surface (1), with that represented by the last equation, will give a line which is a *locus* of singular points: that is to say, every point in the line is a singular point. Such, for instance, is the edge of regression of a developable surface, or the rectilinear directrix of a conoid. There is nothing analogous to this in plane curves, since the one variable is always determined in terms of the other by the equation to the curve, so that there is nothing left indeterminate. In the second place, I observed before that the equation (4) may represent a cone of the second order, or two planes, or a straight line, which last may indeed be considered as a cone, the vertical angle of which is zero. In plane curves, the corresponding equation could represent only two straight lines, or two points, since a homogenous function of two variables can always be decomposed into possible or impossible factors of the first degree. We shall now proceed to some examples in illustration of the general theory given above.

1. Let the equation to the surface be

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 - c^2z^2.$$

This is the locus of the intersection of the tangent planes to the hyperboloid of one sheet, with perpendiculars on them from the centre. Here

$$\begin{array}{lll} U=2x(2r^2-a^2), & V=2y(2r^2-b^2), & W=2z(2r^2+c^2), \\ u=2(2r^2-a^2)+8x^2, & v=2(2r^2-b^2)+8y^2, & w=2(2r^2+c^2)+8z^2, \\ u'=8yz, & v'=8xz & w'=8xy. \end{array}$$

At the origin, or when $x = 0$, $y = 0$, $z = 0$, U, V, W, all vanish; that point is therefore a singular point. Substituting the critical values in the expression for the second differentials, we have

$$\begin{array}{lll} (u) = -2a^2, & (v) = -2b^2, & (w) = 2c^2, \\ (u')=0, & (v')=0, & (w')=0. \end{array}$$

The equation to the tangent cone at the singular point is, therefore,

$$a^2x^2 + b^2y^2 - c^2z^2 = 0.$$

This represents an elliptical cone, the axis of z being perpendicular to the directrix.

2. The equation to Fresnel's Wave Surface is

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(a^2+c^2)y^2 \\ -c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

In this case,

$$U=2x\{a^2(r^2-b^2-c^2)+a^2x^2+b^2y^2+c^2z^2\},$$
$$V=2y\{b^2(r^2-a^2-c^2)+a^2x^2+b^2y^2+c^2z^2\}.$$
$$W=2z\{c^2(r^2-a^2-b^2)+a^2x^2+b^2y^2+c^2z^2\}.$$

Now these expressions vanish when

$$y = 0, \quad r^2 = b^2,$$

involving as values of x and z

$$x = \pm c\sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad z = \pm a\sqrt{\frac{b^2-c^2}{a^2-c^2}}.$$

There are therefore four singular points corresponding to the four different ways in which the double signs of x and z may be combined. We find also

$$(u)=8a^2c^2\frac{a^2-b^2}{a^2-c^2}, \quad (v)=-2(a^2-b^2)(b^2-c^2), \quad (w)=8a^2c^2\frac{b^2-c^2}{a^2-c^2},$$

$$(u')=0, \quad (v')=4ac\sqrt{(a^2-b^2)(b^2-c^2)}\,\frac{a^2+c^2}{a^2-c^2}, \quad (w')=0.$$

Substituting these values in equation (4), and putting the result in the simplest form, we find

$$\frac{x^2}{b^2-c^2}-\frac{a^2-c^2}{4a^2c^2}y^2+\frac{z^2}{a^2-b^2}+\frac{a^2+c^2}{\sqrt{(a^2-b^2)(b^2-c^2)}}\,\frac{xz}{ac}=0,$$

as the equation to the tangent cone at each of the singular points. It is supposed throughout that a, b, c, are in order of magnitude.

3. Let the surface be the locus of the intersection of tangent planes to an elliptical paraboloid, with the perpendiculars on them from the vertex, the equation to it being

$$z(x^2+y^2+z^2)+ax^2+by^2=0.$$

Here $\mathrm{U}=2x(z+a)$, $\mathrm{V}=2y(z+b)$, $\mathrm{W}=x^2+y^2+3z^2$.

These expressions vanish when $x=0$, $y=0$, $z=0$; which values also satisfy the given equation: the origin is therefore a singular point. We find also

$$(u)=2a,\ (v)=2b,\ (w)=0,\ (u')=0,\ (v')=0,\ (w')=0.$$

The equation to the locus of the tangent lines is

$$ax^2+by^2=0.$$

As a and b are both positive, this can be satisfied only by

$$x=0,\ y=0;$$

and it therefore represents the axis of z. It will be seen from the equation that z can never be positive since that would render x and y impossible; the point in question is therefore a cusp. The surface surrounds the negative axis of z which it touches at the origin, so that the form of the surface resembles the shape of the flower of the convolvulus. A correct idea of the surface may be formed by supposing a cissoid to revolve round its axis;—the surface generated is the same as that of the surface here discussed when $a=b$. If a and b be of contrary signs, in which case the surface is formed from the hyperbolic paraboloid, the equation to the locus of the tangent lines is

$$ax^2-by^2=0,$$

which represents two planes perpendicular to the plane of x, y.

4. The equation to the Cono-cuneus of Wallis is

$$a^2y^2-x^2(c^2-z^2)=0.$$

Here

$$\mathrm{U}=-2x(c^2-z^2),\quad \mathrm{V}=2a^2y,\quad \mathrm{W}=2x^2z.$$

These all vanish when $x=0$, $y=0$, independently of any value of z; so that the axis of z is a locus of singular points or a singular line. We find also

$$(u)=-2(c^2-z^2),\quad (v)=2a^2,\quad (w)=0.$$
$$(u')=0,\qquad (v')=0,\quad (w')=0,$$

so that the equation to the locus of the tangent lines is

$$a^2y'^2-(c^2-z^2)x'^2=0,$$

the current coordinates being accentuated to distinguish them from z, the undetermined coordinate of the point under consideration. So long as $z<c$ this equation represents two planes perpendicular to the plane of $x\,y$. z cannot be greater than c, as y would then become impossible.

5. Let the surface to the *héliçoide développable*, the equation to which is

$$x \sin\left(\frac{2\pi}{h}z-\frac{\sqrt{(x^2+y^2-a^2)}}{a}\right)+y \cos\left(\frac{2\pi}{h}z-\frac{\sqrt{(x^2+y^2-a^2)}}{a}\right)=a.$$

If we assume

$$\frac{2\pi}{h}z-\frac{\sqrt{(x^2+y^2-a^2)}}{a}=\theta,$$

we find

$$U=\sin\theta-\frac{x(x\cos\theta-y\sin\theta)}{a\sqrt{(x^2+y^2-a^2)}},$$

$$V=\cos\theta-\frac{y(x\cos\theta-y\sin\theta)}{a\sqrt{(x^2+y^2+a^2)}},$$

$$W=\frac{2\pi}{h}(x\cos\theta-y\sin\theta).$$

From the equation to the surface it is easily seen that

$$x\cos\theta-y\sin\theta=\sqrt{(x^2+y^2-a^2)}.$$

Hence if we assume

$$x=a\sin\frac{2\pi z}{h}, \qquad y=a\cos\frac{2\pi z}{h},$$

the three preceding expressions will vanish, and therefore the line determined by these equations, and the equation to the surface, is a locus of singular points. This line is the intersection of the surface by the cylinder

$$x^2+y^2=a^2,$$

and is evidently the generating helix. Since in the equation to the surface x^2+y^2 can never be less than a^2, it appears that no part of the surface lies within the helix, which is therefore truly an edge of regression.

On proceeding to the second differential coefficients, and substituting in them the critical values of x and y, we find, retaining only the terms which become infinite from involving $\sqrt{(x^2+y^2-a^2)}$ in the denominator,

$$(u)=-2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (v)=2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (w)=0,$$

$$(u')=\frac{2\pi}{h}\cos\frac{2\pi z}{h}, \quad (v')=\frac{2\pi}{h}\sin\frac{2\pi z}{h},$$

$$(w')=\sin^2\frac{2\pi z}{h}-\cos^2\frac{2\pi z}{h},$$

so that the equation to the locus of the tangent lines is

$$(y'^2-x'^2)\sin nz\cos nz-x'y'(\cos^2 nz-\sin^2 nz)$$

$$+ \text{[illegible]}\, z+y'\cos nz)=0,$$

where $n = \frac{2\pi}{h}$, and the accentuated letters are the current co-ordinates of the tangents, and the unaccentuated z is the undetermined co-ordinate of the point of contact.

This equation may be decomposed into two factors,

$$x' \sin nz + y' \cos nz = 0,$$

$$\text{and} \quad y' \sin nz - x' \cos nz + nz' = 0,$$

which are the equations to two planes.

6. I shall take a single example where it is necessary to proceed to an order of differentiation higher than the second.

Let the equation to the surface be

$$(x^2 + y^2 + z^2)^3 = a^2 (y^2z^2 + x^2z^2 + x^2y^2).$$

It is easy to see that the origin is a singular point; and on making $x = 0$, $y = 0$, $z = 0$, all the differential coefficients of the second and third orders vanish, while of the fourth all vanish except three, which are

$$\frac{d^4F}{dx^2dy^2} = \frac{d^4F}{dx^2dz^2} = \frac{d^4F}{dy^2dz^2} = -\ 4a^2.$$

The equation to the locus of the tangent lines is therefore

$$y^2z^2 + x^2z^2 + x^2y^2 = 0.$$

This is satisfied in three ways only, either by

$$x = 0, \qquad y = 0,$$

$$\text{or by} \quad x = 0, \qquad z = 0,$$

$$\text{or by} \quad y = 0, \qquad z = 0.$$

It therefore represents the three axes, and shows that the surface at the origin touches these lines, forming round each a figure resembling that produced by the revolution of a circular arc round a tangent.

It is needless to multiply examples, more especially as there are few surfaces of a high order which are sufficiently important to require a minute discussion.

V.—ON FOURIER'S EXPANSIONS OF FUNCTIONS IN TRIGONOMETRICAL SERIES.*

The object of the following paper is to endeavour to shew in what cases a function, arbitrary between certain limits, can be developed in a series of cosines, and in what cases in a series of sines. Though the whole subject has been treated very fully by

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* From a Correspondent.

Fourier, yet, as he nowhere directly demonstrates, that a function can be developed in a series of sines or cosines separately, and as, from the want of this direct demonstration, many of his formulæ have been believed to be erroneous, the following paper may be interesting to some readers.

It has been rigorously demonstrated, first, so far as I know, by Fourier and afterwards by Poisson, that

$$\pi fx = \tfrac{1}{2}\int_0^{2\pi} fa\,da + \cos x\int_0^{2\pi} \cos a fa\,da + \dots\dots\dots$$
$$+ \sin x\int_0^{2\pi} \sin a fa\,da + \dots\dots,$$

where fx is completely arbitrary, between the limits $x = 0$ and $x = 2\pi$. Now, putting the first part of this series equal to $\pi\phi x$, and the second to $\pi\psi x$, we have

$$fx = \phi x + \psi x \dots\dots\dots (a).$$

But, since $\sin nx = -\sin n(2\pi - x)$, and $\cos nx = \cos n(2\pi - x)$, it follows that $\psi x = -\psi(2\pi - x)$, and $\phi x = \phi(2\pi - x)$, and hence

$$f(2\pi - x) = \phi x - \psi x \dots\dots (b)$$

Hence, by this equation, and (a), we have

$$\tfrac{1}{2}\{fx + f(2\pi - x)\} = \phi x \dots\dots\dots (c)$$

$$\text{and} \quad \tfrac{1}{2}\{fx - f(2\pi - x)\} = \psi x \dots\dots\dots (d).$$

Now, when x is between 0 and π, fx and $f(2\pi - x)$ are perfectly arbitrary, and independent of one another; and therefore it follows that ϕx and ψx are likewise perfectly arbitrary between the same limits. Now

$$\pi\phi x = \tfrac{1}{2}\int_0^{2\pi} fa\,da + \cos x\int_0^{2\pi} \cos a fa\,da$$
$$+ \cos 2x\int_0^{2\pi} \cos 2a fa\,da + \&c.$$
$$= \tfrac{1}{2}\int_0^{\pi} fa\,da + \cos x\int_0^{\pi} \cos a fa\,da$$
$$+ \cos 2x\int_0^{\pi} \cos 2a fa\,da + \&c.$$
$$+ \tfrac{1}{2}\int_0^{\pi} f(2\pi - a)da + \cos x\int_0^{\pi} \cos a f(2\pi - a)\,da$$
$$+ \cos 2x\int_0^{\pi} \cos 2a f(2\pi - a)\,da + \&c.$$
$$= \tfrac{1}{2}\int_0^{\pi} \{fa + f(2\pi - a)\}\,da + \cos x\int_0^{\pi} \cos a\{fa + f(2\pi - a)\}\,da$$
$$+ \cos 2x\int_0^{\pi} \cos 2a\{fa + f(2\pi - a)\}\,da + \&c.$$

or by (c),

$$\frac{\pi}{2}\phi x = \tfrac{1}{2}\int_0^\pi \phi a\, da + \cos x\int_0^\pi \cos a\,\phi a\, da$$
$$+ \cos 2x\int_0^\pi \cos 2a\,\phi a\, da + \&c.$$

In a similar manner it may be shewn, that

$$\frac{\pi}{2}\psi x = \sin x\int_0^\pi \sin a\,\psi a\, da + \sin 2x\int_0^\pi \sin 2a\,\psi a\, da + \&c.$$

Hence we see, that any function whatever of x, may be represented between the limits 0 and π, by a series either of cosines or sines of multiples of x. If it be represented by a series of cosines, then the value of the function corresponding to any value of x, will be found from its values between the limits $x = 0$ and $x = \pi$, by the condition,

$$fx = f(2\pi - x):$$

while if it be represented by a series of sines, the condition will be

$$fx = -f(2\pi - x).$$

The same results might have been obtained by taking, in the original equation, fx subject to either of these conditions. If it be subject to the former, $\int_0^{2\pi} fa \sin na\, da$ will vanish, and fx will be expanded in a series of cosines. If it be subject to the second condition $\int_0^{2\pi} fa \cos na\, da$ will vanish, and fx will be expanded in a series of sines. The two series thus obtained will be equal to one another, when x is between 0 and π; but when x is between π and 2π, the value of one will be the negative of the value of the other.

Fourier gives many expansions of functions in series of sines or cosines alone, obtained from the formulæ given above; which, however he demonstrates merely by assuming fx to be developed in such a series, and then determining the coefficients. Now with regard to these series, Mr. Kelland, in his excellent Treatise on Heat, remarks, that they are "nearly all erroneous." This remark has probably arisen from finding that they differ from the developments obtained from the general formula for functions which follow the given assumption through their whole periods from 0 to 2π, instead of following it merely between the limits 0 and π, as is the case if sines or cosines alone be used.

Thus, Mr. Kelland gives the following expansion of a function which is equal to c, when x is between 0 and a, and to zero, when x is between a and 2π:

$$\pi\frac{\phi x}{c} = \tfrac{1}{2}a + \sin a \cos x + \tfrac{1}{2}\sin 2a \cos 2x + \ldots\ldots$$
$$+ \operatorname{vers} a \sin x + \tfrac{1}{2}\operatorname{vers} 2a \sin 2x + \ldots\ldots;$$

an expression differing, as he remarks, from Fourier's, "which em-

braces only the second line of this." Now, as long as x lies between 0 and π, the two series of which this expression is composed, are equal to one another, if we suppose, for the present, $a < \pi$, and their sum is the required function between these limits. When, however, x is between π and 2π, the value of the one series is the negative of the value of the other, as is readily seen from what has been said above. Hence, when x is between π and 2π the value of the expression is nothing. Now Fourier proposes to find an expansion of a function which is equal to c, when x is between 0 and a, and to zero when x is between a and π, without any supposition regarding the values of the function, when x is between π and 2π. Now it is clear, that the double of either of the series in the expansion given by Mr. Kelland, will be sufficient for this purpose, and Fourier, "Théorie de la chaleur," page 244, chooses to make use of the second; that is, he develops the required function in a series of sines of multiples of x.

The series given by Fourier may be verified in the following manner:

$$\begin{aligned} \text{Let } u &= \text{vers}\, a \sin x + \tfrac{1}{2}\, \text{vers}\, 2a \sin 2x + \&c. \\ &= \sin x + \tfrac{1}{2} \sin 2x + \&c. \\ &\quad - \tfrac{1}{2}\{\sin(a+x) + \tfrac{1}{2}\sin 2(a+x) + \&c.\} \\ &\quad + \tfrac{1}{2}\{\sin(a-x) + \tfrac{1}{2}\sin 2(a-x) + \&c.\} \end{aligned}$$

Now it is well known, and it is demonstrated by Mr. Kelland, p. 59, that

$$\sin\theta + \tfrac{1}{2}\sin 2\theta + \&c. = \tfrac{1}{2}(\pi - \theta),$$

which obviously holds for any value of θ between 0 and 2π, and for no others. Hence, x and a being, of course, each less than π, we have, when $x < a$,

$$u = \tfrac{1}{2}\pi.$$

When $x > a$, the last series may be put under the form

$$-\tfrac{1}{2}\{\sin(x-a) + \tfrac{1}{2}\sin 2(x-a) + \&c.\},$$

and consequently we have

$$u = 0.$$

The values of u are thus found for all values of a and x between 0 and π; and, from them, those for any values of a and x are readily found.

In exactly a similar manner it may be shown, that the series given by Mr. Kelland, p. 64, is really the expansion of a very different function from the one of which the series given by Fourier, p. 246, is the expansion. Thus, the trapezium represented by Mr. Kelland's series occupies the whole space along the axis of x, from 0 to 2π; while, in that space, two trapeziums are represented by Fourier's, one extending from 0 to π, and the other from π to 2π, and inverted, with regard to the axes both of x and y. It may also be remarked, that the form of the trapezium represented by Mr. Kelland's series is much more general than that of the trapezium represented by Fourier's.

Fourier's series may be verified in the following manner:

$$\text{Let } u = \frac{4}{\pi}(\sin a \sin x + \frac{1}{3^2} \sin 3a \sin 3x + \ldots).$$

$$\text{Then } \frac{du}{dx} = \frac{4}{\pi}(\sin a \cos x + \frac{1}{3} \sin 3a \cos 3x + \ldots).$$

$$= \frac{2}{\pi}\left\{\begin{array}{l}\sin (a + x) + \frac{1}{3} \sin 3(a + x) + \ldots \\ + \sin (a - x) + \frac{1}{3} \sin 3(a - x) + \ldots\end{array}\right\}$$

Now, if a and x be each less than $\frac{1}{2}\pi$, we have (Fourier, p. 237, or Kelland, p. 65,)

$$\frac{du}{dx} = 1, \quad \text{when } x < a,$$

$$\text{and } \frac{du}{dx} = 0, \quad \text{when } x > a.$$

Hence we have, obviously,

$$u = x, \quad \text{from } x = 0 \text{ to } x = a,$$

$$\text{and } u = a, \quad \text{from } x = a \text{ to } x = \tfrac{1}{2}\pi.$$

Now, since $\sin (2n + 1) x = \sin (2n + 1)(\pi - x)$, the values of u are the same for x as for $\pi - x$. Hence

$$u = x, \quad \text{from } x = 0 \text{ to } x = a,$$

$$u = a, \quad \text{from } x = a \text{ to } x = \pi - a,$$

$$\text{and } u = \pi - x, \quad \text{from } x = \pi - a \text{ to } x = \pi.$$

I have examined the other series given by Fourier, on this subject, and they seem to be all correct, with the exception of misprints and mistakes in transcription, which, unfortunately, are very numerous. The only one of these mistakes which can cause any serious embarrassment, is with regard to the series at the top of p. 245. The value of this series, between the limits $x = 0$ and $x = a$, is $\sin \frac{\pi x}{a}$, and not $\sin x$, as is there stated. The same error occurs in p. 444, where an application of this series is made to determine the value of the definite integral,

$$\int_0^\infty \frac{dq \sin qx \sin q\pi}{1 - q^2};$$

but Fourier must have been in possession of the correct value, as it is that value which he employs in this application.

P. Q. R.

VI.—ON CERTAIN INTEGRAL TRANSFORMATIONS.

By B. Bronwin.

Denby, Oct. 12th, 1840.

Sir,—I transmit you a few Integral Transformations. The integrals are reduced to elliptic, or such as are known to be elliptic functions. I am not aware that they have ever been reduced to such before, with the exception of three or four of them; and the mode of reduction is so simple as to deserve notice. Their form in general is the most useful, and the most conspicuous to serve as way-marks. Many more, however, are wanting to complete the list, so as to include every variation and combination of exponent. If that could be done they would deserve to be tabulated, with the substitutions proper to effect their reduction.

I am, Sir, your obedient servant,

B. BRONWIN.

To the Editor of the Cambridge Mathematical Journal.

In what follows, the letters E, F, denote elliptic functions, or rather functions, which by methods indicated by M. Legendre, are easily reduced to such. The letters a, b, c, &c. indicate the particular transformations to which they are attached.

$$\text{Let} \quad y=\frac{(1+c)\,x}{1+cx^2}, \quad k=\frac{2\sqrt{c}}{1+c} \;\ldots\ldots\; (a).$$

Then

$$\sqrt{(1-y^2)}=\frac{\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}{1+cx^2}, \quad \sqrt{(1-k^2y^2)}=\frac{1-cx^2}{1+cx^2},$$

$$cx^2=\frac{1-\sqrt{(1-k^2x^2)}}{1+\sqrt{(1-k^2x^2)}}=\frac{\{1-\sqrt{(1-k^2y^2)}\}^2}{k^2y^2}, \quad x=\frac{1-\sqrt{(1-k^2y^2)}}{c^{\frac{1}{2}}ky},$$

$$\sqrt{x}=\frac{\sqrt{(1+ky)}-\sqrt{(1-ky)}}{\sqrt{(2c^{\frac{1}{2}}k)}\,.\,\sqrt{y}},$$

$$\frac{1}{\sqrt{x}}=\frac{\sqrt[4]{c}}{\sqrt{(2k)}}\,.\,\frac{\sqrt{(1+ky)}+\sqrt{(1-ky)}}{\sqrt{y}}$$

$$dy=\frac{(1+c)\,(1-cx^2)\,dx}{(1+cx^2)^2},$$

$$\frac{dy}{\sqrt{(1-y^2)}\,.\,\sqrt{(1-k^2y^2)}}=\frac{(1+c)\,dx}{\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}.$$

From this last, by means of the equations preceding, we easily deduce the four following:

$$\frac{2\sqrt{c}\,\sqrt{(1+c)}\,.\,\sqrt{x}\,dx}{\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}$$

$$=\frac{dy}{\sqrt{(1-y^2)}\,\sqrt{(y-ky^2)}}-\frac{dy}{\sqrt{(1-y^2)}\,.\,\sqrt{(y+ky^2)}} \;\ldots\ldots (1),$$

$$\frac{2\sqrt{(1+c)}\,.\,dx}{\sqrt{x}\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}$$
$$=\frac{dy}{\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}}+\frac{dy}{\sqrt{(1-y^2)}.\sqrt{1+ky^2)}}\ldots\ldots(2),$$

$$\frac{2ck\sqrt{(1+c)}.x^{\frac{3}{2}}dx}{\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}$$
$$=\frac{(2-ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}}-\frac{(2+ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(1+ky^2)}}\ldots\ldots(3),$$

$$\frac{2k\sqrt{\left(\frac{1+c}{c}\right)}\,.\,dx}{x\sqrt{x}\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}}$$
$$=\frac{(2-ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}}+\frac{(2+ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(y+ky^2)}}\ldots\ldots(4).$$

These are all E, F. We now proceed to another transformation.

$$\text{Let}\quad y=\frac{\sqrt{(1-x^2)}}{\sqrt{(1-c^2x^2)}},\quad\text{or}\quad x=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-c^2y^2)}}\ \ldots\ldots\ (b).$$

$$\text{Then}\quad \sqrt{(1-c^2x^2)}=\frac{\sqrt{(1-c^2)}}{\sqrt{(1-c^2y^2)}},\quad \sqrt{(1-x^2)}=\frac{y\sqrt{(1-c^2)}}{\sqrt{(1-c^2y^2)}},$$

$$\frac{x\sqrt{(1-c^2)}}{\sqrt{(1-x^2)}}=\frac{\sqrt{(1-y^2)}}{y};$$

$$dy=\frac{-(1-c^2)\,x\,dx}{(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{3}{2}}},$$

$$\frac{dy}{\sqrt{(1-y^2)}.\sqrt{(1-c^2y^2)}}=\frac{-dx}{\sqrt{(1-x^2)}.\sqrt{(1-c^2x^2)}}.$$

In the same manner as before we may deduce from these the following:

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{-\sqrt{y}\,.\,dy}{\sqrt{(1-y^2)}\,.\,\sqrt{(1-c^2y^2)}}\ldots\ldots\ldots(5).$$

The last is an E, F, by (1).

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-dy}{\sqrt{y}\sqrt{(1-y^2)}\,.\,\sqrt{(1-c^2y^2)}}\ldots\ldots(6).$$

This is a E, F, by (2).

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-\sqrt{(1-c^2)}.\sqrt{y}\,dy}{(1-c^2y^2)\sqrt{(1-y^2)}}\ldots\ldots\ldots\ldots(7),$$

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{-dy}{\sqrt{(1-c^2)}\cdot\sqrt{y}\sqrt{(1-y^2)}} \;\ldots\ldots\; (8).$$

The two last are plainly E, F.

$$\frac{\sqrt{x}\,dx}{(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{-\sqrt[4]{(1-c^2)}\cdot dy}{(1-y^2)^{\frac{1}{4}}(1-c^2y^2)} \;\ldots\ldots\ldots\;\ldots\ldots\; (9),$$

$$\frac{\sqrt{x}\,dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \frac{-\sqrt[4]{(1-c^2)}\cdot\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-c^2y^2)} \;\ldots\;\ldots\ldots\ldots\; (10).$$

In (9) make $(1-y^2)^{\frac{1}{4}} = v$; and in (10), $\dfrac{\sqrt{y}}{(1-y^2)^{\frac{1}{4}}} = \dfrac{1}{v}$, or $\left(\dfrac{1}{y^2}-1\right)^{\frac{1}{4}} = v$; and these are both evidently E, F.

$$\frac{\sqrt[4]{(1-c^2)}\cdot dx}{\sqrt{x}\,(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \frac{-dy}{\sqrt{y}\,(1-y^2)^{\frac{3}{4}}} \;\ldots\ldots\ldots\;\ldots\ldots\; (11).$$

If we make $y = \dfrac{1-v^2}{1+v^2}$, this is seen to be an E, F.

Next, let $y = \left(\dfrac{1+c}{2}\right)^{\frac{1}{2}} \dfrac{\sqrt{(1+x)}}{\sqrt{(1+cx)}}$, $\quad k = \dfrac{2\sqrt{(c)}}{1+c} \;\ldots\ldots\; (c).$

Then $\sqrt{(1-y^2)} = \left(\dfrac{1-c}{2}\right)^{\frac{1}{2}} \cdot \dfrac{\sqrt{(1-x)}}{\sqrt{(1+cx)}}$,

$$\sqrt{(1-k^2y^2)} = \left(\frac{1-c}{1+c}\right)^{\frac{1}{2}} \cdot \frac{\sqrt{(1-cx)}}{\sqrt{(1+cx)}};$$

$$\frac{dy}{\sqrt{(1-y^2)}\cdot\sqrt{(1-k^2y^2)}} = \frac{1+c}{2}\,\frac{dx}{\sqrt{(1-x^2)}\cdot\sqrt{(1-c^2x^2)}}.$$

From hence we derive, by the same process as before,

$$\frac{dx}{(1-x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+x)^{\frac{1}{4}}(1+cx)^{\frac{3}{4}}} = \left(\frac{2}{1+c}\right)^{\frac{5}{4}} \frac{\sqrt{y}\,dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \;\ldots\ldots\ldots\; (12),$$

$$\frac{dx}{(1-x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+x)^{\frac{3}{4}}(1+cx)^{\frac{1}{4}}} = \left(\frac{2}{1+c}\right)^{\frac{3}{4}} \frac{dy}{\sqrt{y}\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \;\ldots\ldots\ldots\; (13).$$

The two last are E, F, by (1) and (2) respectively.

$$\frac{dx}{(1-x)^{\frac{1}{4}}(1-cx)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}(1+cx)^{\frac{1}{2}}} = \left(\frac{2}{1+c}\right)^{\frac{5}{4}} \cdot \frac{dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{3}{4}}} \;\ldots\ldots\ldots\; (14),$$

$$\frac{dx}{(1-x)^{\frac{3}{4}}(1-cx)^{\frac{1}{4}}(1+x)^{\frac{1}{2}}(1+cx)^{\frac{1}{2}}} = \left(\frac{2}{1+c}\right)^{\frac{3}{4}} \cdot \frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}} \quad \ldots\ldots\ldots (15).$$

The two last are E, F, by (5) and (6).

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{2}}(1+cx)} = \left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{2}{1+c}\right)^{\frac{5}{4}} \cdot \frac{\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots (16).$$

This is an E, F, by (10).

In all the transformations (c) we may make x negative, or

$$y = \left(\frac{1+c}{2}\right)^{\frac{1}{2}} \cdot \frac{\sqrt{(1-x)}}{\sqrt{(1-cx)}}.$$

These will give us an additional number of E, F. I shall not put them down.

Next, let

$$y = \frac{1+\sqrt{c}}{\sqrt{\{2(1+c)\}}} \cdot \frac{\sqrt{(1+x)}.\sqrt{(1+cx)}}{1+c^{\frac{1}{2}}x}, \quad c = \left\{\frac{1-\sqrt[4]{(1-k^2)}}{1+\sqrt[4]{(1-k^2)}}\right\}^2 \ldots (d).$$

Then $$\sqrt{(1-y^2)} = \frac{1-\sqrt{c}}{\sqrt{\{2(1+c)\}}} \cdot \frac{\sqrt{(1-x)}.\sqrt{(1-cx)}}{1+c^{\frac{1}{2}}x},$$

$$\sqrt{(1-k^2y^2)} = (1-k^2)^{\frac{1}{4}}\frac{1-c^{\frac{1}{2}}x}{1+c^{\frac{1}{2}}x}, \quad c^{\frac{1}{2}}x = \frac{\sqrt[4]{(1-k^2)}-\sqrt{(1-k^2y^2)}}{\sqrt[4]{(1-k^2)}+\sqrt{(1-k^2y^2)}};$$

$$\frac{dy}{\sqrt{(1-y^2)}.\sqrt{(1-k^2y^2)}} = \frac{(1+\sqrt{c})^2}{2} \cdot \frac{dx}{\sqrt{(1-x^2)}.\sqrt{(1-c^2x^2)}}.$$

From these we derive

$$\frac{dx}{(1+x)^{\frac{1}{4}}(1-x)^{\frac{1}{2}}(1+cx)^{\frac{1}{4}}(1-cx)^{\frac{1}{2}}(1+c^{\frac{1}{2}}x)^{\frac{1}{2}}} = \frac{2\sqrt[4]{\{2(1+c)}}{(1+\sqrt{c})^{\frac{5}{2}}} \cdot \frac{\sqrt{y}\,dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \quad \ldots\ldots (17).$$

This is an E, F, by (1) and (2).

$$\frac{dx}{(1+x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}(1+c^{\frac{1}{2}}x)} = \frac{2\sqrt{\{2(1+c)\}}}{(1+\sqrt{c})^2\sqrt{(1-c)}} \cdot \frac{\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}} \quad \ldots\ldots (18),$$

an E, F, by (10).

By this transformation we might obtain more formulæ.

Let $y = \dfrac{\sqrt{(1-x^2)}\,\sqrt{(1-c^2x^2)}}{1-c^2x^2}, \quad k = \dfrac{2\sqrt{c}}{1+c} \;\ldots\ldots\; (e).$

Then $\sqrt{(1-y^2)} = \dfrac{(1-c)\,x}{1-cx^2}, \quad \sqrt{(1-k^2y^2)} = \dfrac{1-c}{1+c}\cdot\dfrac{1+cx^2}{1-cx^2}.$

$$\frac{dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} = \frac{-(1+c)\,dx}{\sqrt{(1-x^2)}\,\sqrt{(1-c^2x^2)}}.$$

Hence we find

$$\frac{(1+c)\,dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}(1-cx^2)^{\frac{1}{2}}} = \frac{-\sqrt{y}\,dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \;\ldots\ldots\; (19),$$

$$\frac{(1+c)\,(1-cx^2)^{\frac{1}{2}}\,dx}{(1-x^2)^{\frac{3}{4}}\,(1-c^2x^2)^{\frac{3}{4}}} = \frac{-dy}{\sqrt{y}\,\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \;\ldots\ldots\ldots\ldots\; (20).$$

VII.—ON A THEOREM IN THE GEOMETRY OF POSITION.*

We propose to apply the following (new?) theorem to the solution of two problems in Analytical Geometry.

Let the symbols

$$\begin{vmatrix} \alpha \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \quad \&c.$$

denote the quantities

$$\alpha, \quad \alpha\beta' - \alpha'\beta, \quad \alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma, \quad \&c.$$

(The law of whose formation is tolerably well known, but may be thus expressed,

$$\begin{vmatrix} \alpha \end{vmatrix} = \alpha, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} = \alpha \begin{vmatrix} \beta' \end{vmatrix} - \alpha' \begin{vmatrix} \beta \end{vmatrix},$$

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \omega', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \alpha \begin{vmatrix} \beta', & \gamma' \\ \beta'', & \gamma'' \end{vmatrix} + \alpha' \begin{vmatrix} \beta'', & \gamma'' \\ \beta, & \gamma \end{vmatrix} + \alpha'' \begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \quad \&c.$$

the signs + being used when the number of terms in the side of the square is odd, and + and − alternately when it is even.)

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* From a Correspondent.

Then the theorem in question is

$$\left|\begin{array}{lll} \rho\alpha+\sigma\beta+\tau\gamma\ldots, & \rho\alpha'+\sigma\beta'+\tau\gamma'\ldots, & \rho\alpha''+\sigma\beta''+\tau\gamma''\ldots \\ \rho'\alpha+\sigma'\beta+\tau'\gamma\ldots, & \rho'\alpha'+\sigma'\beta'+\tau'\gamma'\ldots, & \rho'\alpha''+\sigma'\beta''+\tau'\gamma''\ldots \\ \rho''\alpha+\sigma''\beta+\tau''\gamma\ldots, & \rho''\alpha'+\sigma''\beta'+\tau''\gamma'\ldots, & \rho''\alpha''+\sigma''\beta''+\tau''\gamma''\ldots \\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \end{array}\right|$$

$$= \left|\begin{array}{l} \rho,\ \sigma,\ \tau\ldots \\ \rho',\ \sigma',\ \tau'\ldots \\ \rho'',\ \sigma''\ \tau''\ldots \\ \vdots \quad \vdots \quad \vdots \end{array}\right| \left|\begin{array}{l} \alpha,\ \beta,\ \gamma\ldots \\ \alpha',\ \beta'\ \gamma'\ldots \\ \alpha'',\ \beta'',\ \gamma''\ldots \\ \vdots \quad \vdots \quad \vdots \end{array}\right|$$

(This theorem admits of a generalisation which we shall not have occasion to make use of, and which therefore we may notice at another opportunity.)

To find the relation that exists between the distances of five points in space.

We have, in general, whatever x_1, y_1, z_1, w_1, &c., denote.

$$\left|\begin{array}{cccccc} x_1^2+y_1^2+z_1^2+w_1^2, & -2x_1, & -2y_1, & -2z_1, & -2w_1, & 1. \\ x_2^2+y_2^2+z_2^2+w_2^2, & -2x_2, & -2y_2, & -2z_2, & -2w_2, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2+w_5^2, & -2x_5, & -2y_5, & -2z_5, & -2w_5, & 1 \\ 1 & ,\ 0, & 0, & 0, & 0, & 0 \end{array}\right|$$

multiplied into

$$\left|\begin{array}{cccccc} 1, & x_1, & y_1, & z_1, & w_1, & x_1^2+y_1^2+z_1^2+w_1^2 \\ 1, & x_2, & y_2, & z_2, & w_2, & x_2^2+y_2^2+z_2^2+w_2^2 \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ 1, & x_5, & y_5, & z_5, & w_5, & x_5^2+y_5^2+z_5^2+w_5^2 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{array}\right| =$$

$$\left|\begin{array}{cccccc} \overline{x_1-x_1}^2+\overline{y_1-y_1}^2+\overline{z_1-z_1}^2+\overline{w_1-w_1}^2, & \overline{x_1-x_2}^2+\ldots, & \overline{x_1-x_3}^2+\ldots, & \overline{x_1-x_4}^2+\ldots, & \overline{x_1-x_5}^2+\ldots, & 1 \\ \overline{x_2-x_1}^2+\ldots, & \overline{x_2-x_2}^2+\ldots, & \overline{x_2-x_3}^2+\ldots, & \overline{x_2-x_4}^2+\ldots, & \overline{x_2-x_5}^2+\ldots, & 1 \\ \cdot & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ \cdot & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ \overline{x_5-x_1}^2\ldots, & \overline{x_5-x_2}^2+\ldots, & \overline{x_5-x_3}^2+\ldots, & \overline{x_5-x_4}^2+\ldots, & \overline{x_5-x_5}^2+\ldots, & 1 \\ & 1, & 1, & 1, & 1, & 0 \end{array}\right|$$

Putting the w's equal to 0, each factor of the first side of the equation vanishes, and therefore in this case the second side of the equation becomes equal to zero. Hence $x_1, y_1, z_1, x_2, y_2, z_2$, &c. being the co-ordinates of the points 1, 2, &c. situated arbitrarily in space, and $\overline{12}^2$, $\overline{13}^2$, &c. denoting the distances between these points, we have immediately the required relation

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is easily expanded, though from the mere number of terms the process is somewhat long.

Precisely the same investigation is applicable to the case of four points in a plane, or three points in a straight line. Thus the former gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The latter gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Or expanding,

$$\overline{12}^4 + \overline{13}^4 + \overline{23}^4 - 2\,\overline{12}^2\,\overline{13}^2 - 2\,\overline{13}^2\,\overline{23}^2 - 2\,\overline{12}^2\,\overline{23}^2 = 0;$$

which may be derived immediately, by the equation

$$\pm\,\overline{12} \pm \overline{13} = \pm\,\overline{23},$$

and is the simplest form under which this equation, cleared of the ambiguous sign, can be put.

(The above result may be deduced so elegantly from the general theory of elimination, that notwithstanding its simplicity it is perhaps worth mentioning.)

Let $$x_{\prime\prime}-x_{\prime\prime\prime}=\alpha,\quad x_{\prime\prime\prime}-x_{\prime}=\beta,\quad \overline{x_{\prime}-x_{\prime\prime}}=\gamma;$$

then $\overline{12}^2=\gamma^2$, $\overline{23}^2=\alpha^2$, $\overline{31}^2=\beta^2$, and $\alpha+\beta+\gamma=0$;

from which α, β, γ, are to be eliminated. Multiplying the last equation by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and reducing by the three first,

$$\begin{aligned}
0\,.\,\alpha + \overline{12}^2.\,\beta + \overline{31}^2.\,\gamma + \alpha\beta\gamma &= 0,\\
\overline{12}^2.\,\alpha + 0\,.\,\beta + \overline{23}^2.\,\gamma + \alpha\beta\gamma &= 0,\\
\overline{31}^2.\,\alpha + \overline{23}^2.\,\beta + 0\,.\,\gamma + \alpha\beta\gamma &= 0,\\
\alpha + \beta + \gamma + 0\,.\,\alpha\beta\gamma &= 0;
\end{aligned}$$

from which, eliminating α, β, γ, $\alpha\beta\gamma$ by the general theory of simple equations,

$$\begin{vmatrix}
0, & \overline{12}^2, & \overline{13}^2, & 1\\
\overline{21}^2, & 0, & \overline{23}^2, & 1\\
\overline{31}^2, & \overline{32}^2, & 0, & 1\\
1, & 1, & 1. & 0
\end{vmatrix} = 0.$$

The (additional) equation that exists between the distances of five points on a sphere or four points in a circle, has such a remarkable analogy with the preceding, that they almost require to be noticed at the same time.

If α, β, γ, r be the co-ordinates of the centre, and the radius of the sphere, and
$$\delta=\alpha^2+\beta^2+\gamma^2-r^2,$$
we have immediately
$$\begin{array}{ccccccccccccccc}
x_1^2 &+& y_1^2 &+& z_1^2 &-& 2\alpha x_1 &-& 2\beta y_1 &-& 2\gamma z_1 &+& \delta &=& 0,\\
\vdots && \vdots && \vdots && \vdots && \vdots && \vdots && \vdots && \vdots\\
x_5^2 &+& y_5^2 &+& z_5^2 &-& 2\alpha x_5 &-& 2\beta y_5 &-& 2\gamma z_5 &+& \delta &=& 0;
\end{array}$$
whence eliminating α, β, γ, δ,
$$\begin{vmatrix}
x_1^2+y_1^2+z_1^2, & -2x_1, & -2y_1, & -2z_1, & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
x_5^2+y_5^2+z_5^2, & -2x_5, & -2y_5, & -2z_5, & 1
\end{vmatrix} = 0.$$
Multiplying by
$$\begin{vmatrix}
1, & x_1, & y_1, & z_1, & x_1^2+y_1^2+z_1^2\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
1, & x_5, & y_5, & z_5, & x_5^2+y_5^2+z_5^2
\end{vmatrix}$$

we have immediately

$$\left|\begin{matrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^3, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{matrix}\right| = 0;$$

forming the analogous equation for four points in a circle, and expanding, we readily deduce

$$\overline{14}^4 . \overline{23}^4 + \overline{12}^4\ \overline{34}^4 + \overline{13}^4 . \overline{24}^4 - 2 . \overline{12}^2\ \overline{34}^2\ \overline{13}^2\ \overline{24}^2 - 2 . \overline{14}^2\ \overline{23}^2\ \overline{13}^2,$$

$$\overline{24}^2 - 2 . \overline{14}^2\ \overline{23}^2\ \overline{12}^2\ \overline{34}^2 = 0,$$

which is the rational, and therefore analytically the most simple form of

$$\overline{12}\ \overline{34} + \overline{14}\ \overline{23} = \overline{13}\ \overline{24}.$$

Euclid, B. vi., last proposition.

(It may be remarked that the two factors we have employed in the preceding eliminations, only differ by a numerical factor.)

C.

VIII.—ANALYTICAL DEMONSTRATIONS OF DR. MATTHEW STEWART'S THEOREMS.

By R. Leslie Ellis, B.A., Fellow of Trinity College.

In 1746, Dr. Matthew Stewart, the father of Dugald Stewart, published his "General Theorems." He was at that time a candidate for the chair of mathematics at Edinburgh, then vacant by the death of Maclaurin; and his success is attributed to the celebrity which these remarkable propositions immediately acquired. They were enunciated by Dr. Stewart without demonstrations, and remained undemonstrated till 1805. Mr. Glenie, in the *Edinburgh Transactions* for that year, has given a geometrical method by which the General Theorems and other similar results may be established.

But as yet they have not, I believe, been proved, except by Geometry; and in an article in the 17th volume of the *Edinburgh Review*, ascribed to Playfair, they are strongly recommended to

the attention of analysts. It is hoped, therefore, that the following attempt will have some degree of interest.

We shall begin by establishing a general proposition, from which all the theorems in question, and many others, may be deduced.

LEMMA. If $f\phi$ is a rational and integral function of $\sin\phi$ and $\cos\phi$, then a value may always be assigned to n, such that

$$f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \ldots f\left(\phi + \frac{n-1}{n}2\pi\right)$$

shall be independent of ϕ.

The preceding expression is equivalent to

$$(1 + \mathrm{D} + \ldots \mathrm{D}^{n-1})f\phi, \quad \left(\text{where } \mathrm{D}\phi = \phi + \frac{2\pi}{n}\right),$$

and therefore to

$$\frac{\mathrm{D}^n f\phi - f\phi}{\mathrm{D}-1} = \Delta^{-1}\{f(\phi + 2\pi) - f\phi\} = \Delta^{-1}0.$$

Now $\Delta^{-1}0 = \Sigma a_m \sin mn\phi + \Sigma b_m \cos mn\phi$, ($m$ integral).

Hence $f(\phi) + \ldots + f\left(\phi + \dfrac{n-1}{n}2\pi\right) = \Sigma a_m \sin mn\phi + \&c.$

Let the index of the highest power of $\sin\phi$ or $\cos\phi$ in $f\phi$ be p; then it is easily seen that when $f\phi$ is developed, as it may always be, in a series of sines and cosines of the multiple arcs, $p\phi$ will be the largest arc that can enter into the development. But if n is greater than p, $mn\phi$ will be greater than $p\phi$, except when m is zero. Hence the development

$$\Sigma a_m \sin mn\phi + \&c.$$

cannot coincide with that obtained by summing the separate developments of $f\phi$. $f\left(\phi + \dfrac{2\pi}{n}\right)$, &c., unless a_m and b_m are $=0$ in every case, except when $m = 0$. Hence as $\sin mn\phi = 0$ when $m = 0$, the expression will be reduced to b_0, and we shall have, when $n > p$,

$$f(\phi) + \ldots f\left(\phi + \frac{n-1}{n}2\pi\right) = b_0 \ldots \text{a constant.}$$

Q. E. D.

The constant b_0, will of course be the sum of the constant parts of the developments of $f\phi$, &c.; and as these are all equal and are n in number, it will be n times the constant term in $f\phi$. Now by Fourier's theorem this is equal to $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\phi \,.\, d\phi$, as indeed is obvious. Hence

$$f\phi + \ldots f\left(\phi + \frac{n-1}{n}2\pi\right) = \frac{n}{2\pi}\int_0^{2\pi} f\phi\, d\phi, \text{ when } n > p,$$

which is our fundamental formula.

The first of Stewart's propositions is the following:

From any point in the circumference of a circle draw perpendiculars p, p_1, &c. to the sides of a regular n-sided polygon circumscribed about it; then, if r is the radius,

$$2\Sigma p^3 = 5nr^3 \ldots\ldots \ldots\ldots (1) \ldots\ldots\ldots n > 3.$$

DEM. Let the assumed point subtend at the centre an angle ϕ from the adjacent point of contact. Then

$$p = r(1-\cos\phi), \quad p_1 = r\left\{1 - \cos\left(\phi + \frac{2\pi}{n}\right)\right\}, \text{ \&c. \&c.}$$

$$\therefore \ \Sigma p^3 = r^3 \Sigma (1 - \cos\phi)^3;$$

and by the general formula,

$$\Sigma (1-\cos\phi)^3 = \frac{n}{2\pi}\int_0^{2\pi} (1-\cos\phi)^3 \, d\phi \ \ldots\ldots \ n \text{ being } > 3.$$

$$\text{Now} \quad \int_0^{2\pi} (1-\cos\phi)^3 \, d\phi = 2^4 \int_0^{\pi} \sin^6\theta \, d\theta \, . . (\theta = \tfrac{1}{2}\phi)$$

$$\text{and} \quad \int_0^{\pi} \sin^6\theta \, d\theta = \frac{5.3.1}{6.4.2} \cdot \pi = \frac{5}{2^5} 2\pi;$$

$$\therefore \ \Sigma (1-\cos\phi)^3 = \frac{5n}{2},$$

and therefore $2\Sigma p^3 = 5nr^3$. Q. E. D.

This is a particular case of the second proposition in which the assumed point is not confined to the circumference of the circle, but may have any position whatever. Let l be its distance from the centre; then

$$2\Sigma p^3 = 2nr^3 + 3nl^2 r \ldots n > 3 \ldots \ \ldots\ldots (2).$$

DEM. In this case $p = r - l\cos\phi$, &c. = &c.

$$\therefore \ \Sigma p^3 = nr^3 - 3r^2 l \, \Sigma\cos\phi + 3rl^2 \, \Sigma\cos^2\phi - l^3 \, \Sigma\cos^3\phi.$$

$$\text{But} \int_0^{2\pi} \cos\phi \, d\phi = 0, \quad \int_0^{2\pi} \cos^2\phi \, d\phi = \pi, \quad \int_0^{2\pi} \cos^3\phi \, d\phi = 0;$$

$$\therefore \ \Sigma\cos\phi = 0, \quad \Sigma\cos^2\phi = \frac{n}{2}, \quad \Sigma\cos^3\phi = 0,$$

and $2\Sigma p^3 = 2nr^3 + 3nl^2 r$. Q. E. D.

In the third proposition, a regular n-sided polygon is *inscribed* in the circle, and lines c, c_1, &c. are drawn from its corners to a point assumed in the circumference; then

$$\Sigma c^4 = 6nr^4 \ldots\ldots \ldots (3).$$

DEM. The assumed point and adjacent corner subtending an angle ϕ at the centre, we have

$$c^2 = 2r^2 (1 - \cos\phi$$

$$\therefore\ \Sigma c^4 = 4r^4\, \Sigma (1 - \cos\phi)^2,$$

$$\int_0^{2\pi} (1 - \cos\phi)^2\, d\phi = 2^3 \cdot \frac{3\cdot 1}{4\cdot 2}\pi = 2\pi\,\frac{3}{2};$$

$$\therefore\ \Sigma (1 - \cos\phi)^2 = \frac{3n}{2};$$

$$\therefore\ \Sigma c^4 = 6nr^4. \qquad \text{Q. E. D.}$$

The fourth proposition includes the third. The assumed point may now have any position we please. Let l be its distance from the centre. Here we have

$$c^2 = r^2 + l^2 - 2rl\cos\phi,$$

and $\Sigma c^4 = n(r^4 + l^4 + 2r^2l^2) - 4rl(r^2 + l^2)\,\Sigma\cos\phi + 4r^2l^2\,\Sigma\cos^2\phi.$

By the values above given for $\Sigma\cos\phi$ and $\Sigma\cos^2\phi$, this becomes

$$\Sigma c^4 = nr^4 + 4nr^2l^2 + nl^4 \ldots\ldots (4),$$

which is the proposition in question.

In the fifth proposition we return to the circumscribed polygon, and our object is to determine the sum of the fourth powers of the perpendiculars. As before,

$$\Sigma p^4 = r^4\, \Sigma (1 - \cos\phi)^4,$$

and $$\int_0^{2\pi} (1 - \cos\phi)^4\, d\phi = 2^5\,\frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\pi = 2\pi\cdot\frac{35}{8}.$$

therefore $8\,\Sigma p^4 = 35n\cdot r^4 \ldots\ldots\ldots (5).$ Q. E. D.

In the general case, when l is the distance of the assumed point from the centre,

$$\Sigma p^4 = nr^4 - 4r^3l\,\Sigma\cos\phi + 6r^2l^2\,\Sigma\cos^2\phi - 4rl^3\,\Sigma\cos^3\phi + l^4\,\Sigma\cos^4\phi,$$

and $$\Sigma\cos^4\phi = \frac{n}{2}\cdot\frac{3\cdot 1}{4\cdot 2} = \frac{3n}{8};$$

$$\therefore\ \Sigma p^4 = nr^4 + 6r^2l^2\cdot\frac{n}{2} + \frac{3}{8}nl^4,$$

$$\text{or } 8\,\Sigma p^4 = 8nr^4 + 24nr^2l^2 + 3nl^4 \ldots\ldots (6).$$

This is the sixth proposition.

The seventh is, for the m^{th} power of the perpendiculars, what the first and fifth are for the 3^{d} and 4^{th} powers respectively. It is this:

$$\Sigma p^m = n\,\frac{2m-1\cdot 2m-3\ldots 1}{m\cdot m-1\ldots 1}\,r^m \ldots\ldots (7) \ldots\ldots (n > m).$$

DEM. $$\Sigma p^m = r^m\,\frac{n}{2\pi}\int_0^{2\pi} (1 - \cos\phi)^m\, d\phi.$$

Let $\theta = \frac{1}{2}\phi$;

$$\therefore \int_0^{2\pi} (1-\cos\phi)^m \, d\phi = 2^{m+1} \int_0^{\pi} \sin^{2m}\theta \, d\theta$$

$$= 2^{m+1} \frac{2m-1 \,.\, 2m-3 \ldots 1}{2m \,.\, 2m-2 \ldots 2} \pi = 2\pi \,.\, \frac{2m-1 \ldots 1}{m \,.\, m-1 \ldots 1}$$

$$\therefore \; \Sigma p^m = n \, \frac{2m-1 \,.\, 2m-3 \ldots 1}{m \,.\, m-1 \ldots 1} \, r^m.$$

Q. E. D.

If the assumed point is at l distance from the centre,

$$\Sigma p^m = nr^m - \frac{m}{1} \,.\, r^{m-1} l \, \Sigma \cos\phi + \frac{m \,.\, m-1}{1 \,.\, 2} \, r^{m-2} \, l^2 \, \Sigma \cos^2\phi - \&c.$$

This, it is easily seen, will reduce into the following form

$$\Sigma p^m = nr^m + n \, \frac{m \,.\, m-1}{1.2} \, \frac{1}{2} \, r^{m-2} \, l^2 + n \, \frac{m \ldots m-3}{1.2.3.4} \, \frac{3.1}{4.2} \, r^{m-4} \, l^4 + \&c. \quad \ldots\ldots (8),$$

which is the eighth proposition $\ldots . (n > m)$.

Lastly, let us consider the $2m^{\text{th}}$ powers of the chords in the case of the inscribed polygon: we have already in the third proposition found the value of their sum when $m = 2$.

As before, $\quad c^2 = 2r^2 (1 - \cos\phi)$;

$$\therefore \; \Sigma c^{2m} = 2^m r^{2m} \, \Sigma (1 \quad \cos\phi)^m.$$

That is, as we have already seen,

$$\Sigma c^{2m} = n \, \frac{2m-1 \,.\, 2m-3 \ldots 1}{m \,.\, m-1 \ldots 1} \, r^{2m} \ldots \ldots\ldots (9).$$

We have thus gone through Dr. Stewart's properties of the circle, and have arrived at his results by a simple and uniform method.

It is evident that there is no limit to the number of geometrical theorems which may be deduced from the general formula: almost every curve will afford *interpretations*, if the word may be so used, of our analytical conclusions.

Thus in the ellipse: If any n radii vectores be drawn from the centre at equal angles to one another, the sum of the squares of their reciprocals is equal to n times the square of the reciprocal of that radius vector which is equally inclined to the major and minor axes. For we have

$$\frac{1}{r^2} = \frac{1}{b^2} (1 - e^2 \cos^2\phi);$$

$$\therefore \; \Sigma \frac{1}{r^2} = \frac{n}{b^2} - \frac{e^2}{b^2} \Sigma \cos^2\phi = n \, \frac{1}{b^2} (1 - \tfrac{1}{2} e^2),$$

and $\frac{1}{2} = \cos^2 \frac{\pi}{4}$;

$$\therefore\ \Sigma \frac{1}{r^2} = \frac{n}{b^2}\left(1 - e^2 \cos^2 \frac{\pi}{4}\right),$$

therefore &c. Q. E. D.

It is to be regretted that we have hardly any idea by what considerations Dr. Stewart was led to the curious theorems which bear his name. It is said, indeed, that he was engaged on geometrical porisms when he discovered them, and we are told that he would have published them under the title of porisms, but for his unwillingness to interfere with a subject which the researches of his friend, Dr. Simson, seemed to have appropriated. Whether they are in reality porismatic, is a question on which it would not be worth while to enter.

The fundamental formula of our analysis is perhaps not new; the geometrical applications which we have made of it appear to be original.

IX.—ON A METHOD OF ALGEBRAIC ELIMINATION.*

It is the object of this paper to explain a method of elimination common to algebraic equations, and also to differential equations of all orders and degrees. When applied to the former class, it will always, I believe, lead to calculations which do not differ much from those required by the method in common use; but in principle it appears to me much more simple and satisfactory than that method.

Let it be required to eliminate x from the equations

$$x^2 + px + q = 0,$$
$$x^2 + p'x + q' = 0.$$

Multiply each of the proposed equations by x, and you obtain

$$x^3 + px^2 + qx = 0,$$
$$x^3 + p'x^2 + q'x = 0.$$

These two combined with the two given equations make a system of four equations containing three quantities to be eliminated, viz. x, x^2, x^3; and they are of the first degree with respect to each of these quantities. We may therefore eliminate x, x^2, and x^3, by the rules for equations of the first degree. The result is

$$pq' - p'q + \frac{(q - q')^2}{p - p'} = 0.$$

The same result may be obtained rather more simply thus.

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* From a Correspondent.

Subtract the second equation from the first, and you obtain

$$(p-p')\,x + q - q' = 0.$$

Multiply this equation by x, and the first by $p - p'$, then subtract, and the resulting equation is

$$\{p\,(p - p') - (q - q')\}\,x + q\,(p - p') = 0.$$

Eliminate x between this and the other equation of the first degree; and you obtain

$$(q-q')\,\{p\,(p-p')-(q-q')\}-q\,(p-p')^2=0,$$

or $$(p-p')\,\{p\,(q-q')-q\,(p-p')\}-(q-q')^2=0.$$

or, as before, $$p'q-pq'-\frac{(q-q')^2}{p-p'}=0.$$

It is evident that the methods applied to this example may be employed in treating equations of any degree, and the following rules express both the principle and the method of eliminating one unknown quantity between two algebraic equations which exceed the first degree.

First, suppose the two equations to be of the same degree; and let x denote the quantity to be eliminated, and n the degree of the equations.

Rule. "Multiply each equation by x, n times successively. You thus obtain $2n$ new equations Combine these with whichever you please of the proposed equations, and you have $2n+1$ equations, containing $2n$ quantities to be eliminated, viz. x, x^2, $x^3 \ldots x^{2n}$, and being of the first degree with regard to these quantities. From this system of equations eliminate all the $2n$ quantities by the rules for equations of the first degree."

In practice, the following rule, which is founded on the same principle as the preceding, may frequently be found more convenient. "First eliminate x^n between the two proposed equations. Then multiply the equation thus obtained by x, and eliminate x^n between the resulting equation and either (say the first) of the proposed. You will now have two equations of the $(n-1)^{\text{th}}$ degree. Treat these as you did the two proposed equations, and you will obtain two of the $(n-2)^{\text{th}}$ degree. Continue this process till you obtain an equation not at all affected with x."

If the two proposed equations are not of the same degree, let m denote the degree of the first and n that of the second, n being $< m$.

Rule. "Multiply the second equation by x, $m-n$ times successively; and eliminate x^m, x^{m-1}, $x^{m-2} \ldots x^{n+1}$ between the equations thus formed and the first. You will then have two equations, both of the n^{th} degree, which treat by either of the former rules."

It seems unnecessary to give additional examples in illustration of these rules, as every reader can easily supply such as are likely to be most satisfactory to himself. I will therefore proceed at once to shew, that the same principle by which algebraic equations of

the higher degrees are reduced to equations of the first degree, enables us to reduce differential equations of all orders and degrees, and any number of independent variables, to algebraic equations.

Ex. 1. Let it be required to eliminate y from the equations

$$\frac{dx}{dt} + 4x + 3y = t,$$

$$2x + \frac{dy}{dt} + 5y = e^t.$$

(See Vol. I. p. 175, of this Jonrnal.)

Differentiate the first equation with regard to t, and you obtain the following:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3\frac{dy}{dt} = 1.$$

Eliminate $\frac{dy}{dt}$ between this and the second, and the result is

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 6x - 15y = 1 - 3e^t.$$

Eliminate y between this last result and the first equation, and you obtain a final equation altogether free from y, viz.

$$\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = 1 - 3e^t + 5t.$$

Ex. 2. Let it be required to eliminate x from the equations

$$\frac{dx}{dt} + px + q = 0,$$

$$\frac{dx}{dt} + p'x + q' = 0,$$

(p, q, p', and q' being functions of t, but not containing x).

By subtraction,

$$(p - p')\,x + (q - q') = 0.$$

Differentiate this equation with regard to t, and you obtain

$$(p - p')\frac{dx}{dt} + x\frac{d}{dt}(p - p') + \frac{d}{dt}(q - q') = 0.$$

Eliminate $\frac{dx}{dt}$ between this and the first equation, and the result is

$$\left\{p\,(p - p') - \frac{d}{dt}(p - p')\right\}x + q\,(p - p') - \frac{d}{dt}(q - q') = 0.$$

Eliminate x between the two equations which are free from $\frac{dx}{dt}$, and you obtain a final equation altogether free from x, viz.

$$\left\{p(p - p') - \frac{d}{dt}(p - p')\right\}(q - q') - (p - p')\left\{q(p - p') - \frac{d}{dt}(q\ \ q')\right\} = 0,$$

or

$$(p-p')(p'q-pq')-(q-q')\frac{d}{dt}(p-p')+(p-p')\frac{d}{dt}(q-q')=0.$$

The following rules express the general method of eliminating a function from two equations which contain itself and its differential coefficients with regard to one independent variable.

First, suppose the two equations to be of the same order, and let m denote their order, y the function to be eliminated, t the independent variable.

RULE. "Differentiate both equations m times with respect to t. You thus obtain $2m$ new equations; between which and either (say the first) of the proposed you may eliminate the quantities

$$y, \ \frac{dy}{dt}, \ \frac{d^2y}{dt^2}, \ \ldots\ldots \ \frac{d^{2m}y}{dt^{2m}},$$

by the rules for algebraic equations."

(This rule is not new, see Lacroix, *Traité Elem. de Calc. Dif.*, Art. 134, 3rd edit.)

But it may often be more convenient to proceed by successive eliminations in the following manner:

"First, eliminate $\frac{d^my}{dt}$ between the two proposed equations; then differentiate the equation thus obtained, and between the equation which results from this differentiation and either (say the first) of the proposed, again eliminate $\frac{d^my}{dt^m}$. You will now have two equations of the $(m-1)^{\text{th}}$ order. Treat these as you did the two proposed, and you obtain two of the $(m-2)^{\text{th}}$ order; and by repeating this process as often as necessary, you at last obtain an equation altogether free from y.

If the two proposed equations are not of the same order, let m denote the order of the first, and n that of the second, n being $< m$.

RULE. "Differentiate the second equation $(m-n)$ times; and between the equations thus formed and the first of the proposed, eliminate

$$\frac{d^my}{dt^m}, \ \frac{d^{m-1}y}{dt^{m-1}}, \ \frac{d^{m-2}y}{dt^{m-2}}, \ \ldots\ldots \ \frac{d^{n+1}y}{dt^{n+1}};$$

you will then have two equations of the n^{th} order, which may be treated by the former rule."

When the proposed equations contain, besides the function to be eliminated, its differential coefficients with regard to more than one independent variable, the number of auxiliary equations to be formed is much more numerous: but by repeating the process of differentiation sufficiently often, we always obtain at last r

equations than quantities to be eliminated. If the two proposed equations are of the first order, and there are two independent variables, there will be in the proposed equations three quantities to be eliminated, viz. u, $\frac{du}{dx}$, and $\frac{du}{dy}$, (the function being denoted by u, and the independent variables by x and y). If each of the equations be differentiated with respect to x and with respect to y, there will be in all six equations, containing six quantities, to be eliminated, viz. u, $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$, $\frac{d^2u}{dy^2}$. But if we proceed to the third order, by differentiating each of the new equations with regard both to x and y, we shall have in all fourteen equations, containing only ten quantities to be eliminated, viz.

$$u, \frac{du}{dx}, \frac{du}{dy}, \frac{d^2u}{dx^2}, \frac{d^2u}{dx\,dy}, \frac{d^2u}{dy^2}, \frac{d^3u}{dx^3}, \frac{d^3u}{dx^2\,dy}, \frac{d^3u}{dx\,dy^2}, \frac{d^3u}{dy^3}.$$

The particular form of the proposed will, however, frequently render a much smaller number of auxiliary equations sufficient, as in the following example.

Let it be required to eliminate u from the equations

$$c\frac{du}{dx} + \frac{dz}{dx} + a\frac{dz}{dy} + bz = 0,$$

$$\frac{du}{dx} + a\frac{du}{dy} + bu + c'\frac{dz}{dx} = 0.$$

(See Vol. I., p, 180, of this Journal)

Differentiate the first of these with respect to x and y, and the second with respect to x only, and you obtain the following equations:

$$c\frac{d^2u}{dx^2} + \frac{d^2z}{dx^2} + a\frac{d^2z}{dx\,dy} + b\frac{dz}{dx} = 0,$$

$$c\frac{d^2u}{dx\,dy} + \frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2} + b\frac{dz}{dy} = 0,$$

$$\frac{d^2u}{dx^2} + a\frac{d^2u}{dx\,dy} + b\frac{du}{dx} + c'\frac{d^2z}{dx^2} = 0.$$

The three new equations and the first of the proposed contain only three quantities to be eliminated, viz. $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$; these may therefore be eliminated by the methods employed for algebraic equations. As the equations are of the first degree, the process is very simple; the result is

$$(1-cc')\frac{d^2z}{dx^2} + 2a\frac{d^2z}{dx\,dy} + a^2\frac{d^2z}{dy^2} + 2b\frac{dz}{dx} + 2ab\frac{dz}{dy} + b^2z = 0.$$

In the preceding propositions the problem of elimination seems to me to be placed in a simple and luminous point of view. The different cases treated are referred to one common principle, which seems obvious as soon as it is enunciated, and which will probably be found applicable to other classes of equations.

I shall be very glad if this contribution is seen in as favourable a light by the readers of this Journal, and is found practically useful.

A. Q. G. C.

Feb. 8th, 1841.

April 2nd, 1841.

P.S.—The preceding system of elimination was suggested to me by reading an article on Simultaneous Differential Equations, in the fourth number of this Journal. I observed that the method of separating the symbols of operation from those of quantity, employed in that article, is, so far as elimination is concerned, the same in fact as the method given by La Croix. (The method of the separation of the symbols being however applicable to none but equations of the first degree; but having the advantage, where it is applicable, of indicating at once what differentiations are necessary.) This reflection led me to observe the principle on which elimination between two differential equations depends, viz. that whereas the proposed equations contain several functions of the quantity to be eliminated, this difficulty is evaded by *forming new equations.*

The process by which these equations are formed introduces new functions; nevertheless it answers the purpose for which it is employed, because it increases the number of equations still more than that of functions.

This method seemed natural, and properly applicable to the problem; whereas the method employed in treating algebraic equations of the higher degrees had always appeared to me very unsatisfactory, and obscure in principle. The question was thus raised, whether a method similar to that used in treating differential equations might not be discovered for algebraic. This question being once asked, the answer to it was soon found; especially as the paper which had suggested these reflections pointed out an analogy between the processes of Differentiation and Multiplication.

At the time when this occurred to me, and when I sent my article to you, I was entirely ignorant that any other mathematician had been occupied with the subject, and was not aware that there was any known method of elimination between algebraic equations, except that which makes the problem depend on the method of finding the greatest common measure. From Professor Sylvester's interesting paper in your last number, and from his paper in the *Philosophical Magazine* to which you referred me, I find that that gentleman has not only anticipated me in the fundamental idea, but has likewise devised some very ingenious rules for the more expeditious, and even merely mechanical, application of it. Should you, however, think my article worth inserting,

I shall feel obliged by your allowing this postscript to accompany it, which I will conclude with a remark suggested by Mr. Sylvester's last paper.

If we have to eliminate x between the two equations

$$x^n + px^{n-1} + qx^{n-2} \dots \dots + u = 0,$$
$$x^n + p'x^{n-1} + q'x^{n-2} \dots \dots + u' = 0,$$

we may first eliminate x^n by subtraction, and then write the equations in the form

$$x(x^{n-1} + px^{n-2} + qx^{n-3} + \&c.) + u = 0,$$
$$x(x^{n-1} + p'x^{n-2} + q'x^{n-3} + \&c.) + u' = 0,$$

and eliminate the unbracketed x between these. We thus obtain two equations of the $(n-1)^{\text{th}}$ degree; and in obtaining them we treat the two proposed equations *in the same manner*, which seems preferable to the other methods.

X.—NOTE ON THE DEFINITE INTEGRAL $\int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta$.

The value of $\int_0^{\frac{\pi}{2}} \log \sin\theta\, d\theta$, obviously the same as that of $\int_0^{\frac{\pi}{2}} \log\cos\theta\, d\theta$, was first assigned by Euler, and may be obtained in the following manner.

By Cotes's theorem,

$$z^{2m} - 1 = (z^2-1)\left(z^2 - 2z\cos\frac{1}{m}\pi + 1\right)\dots\left(z^2 - 2z\cos\frac{m-1}{m}\pi + 1\right) \quad \dots\dots (1).$$

Let $z = 1$, then

$$m = 2^{2(m-1)} \sin^2\frac{1}{m}\frac{\pi}{2} \dots\dots \sin^2\frac{m-1}{m}\frac{\pi}{2}.$$

Take the logarithms of both sides, and divide by m, then

$$\frac{\log m + 2(m-1)\log\frac{1}{2}}{2m} = \left(\log\sin\frac{1}{m}\frac{\pi}{2} + \dots + \log\sin\frac{m-1}{m}\frac{\pi}{2}\right)\frac{1}{m}.$$

Let m become infinite, and $= \frac{1}{dx}$: the first side becomes equal

to $\log \frac{1}{2}$, for $\left(\frac{\log m}{m}\right) = 0$ when $m = \frac{1}{0}$; and the second is transformed into the definite integral $\int_0^1 \log \sin x \frac{\pi}{2} \,.\, dx$; therefore $\int_0^1 \log \sin x \frac{\pi}{2}\, dx = \log \frac{1}{2}$.

Let $\theta = x \frac{\pi}{2}$;

$$\therefore \int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta = \frac{\pi}{2} \log \tfrac{1}{2} \,\ldots\ldots\ldots\, (1).$$

Cor. 1. Integrating by parts, we get

$$\int \log \sin \theta \, d\theta = \theta \log \sin \theta - \int \frac{\theta \, d\theta}{\tan \theta}.$$

The integrated part vanishes at both limits,

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\theta \, d\theta}{\tan \theta} = \frac{\pi}{2} \log 2 \,\ldots\ldots\ldots\ldots\, (2).$$

Cor. 2. Let $\sin \theta = e^{-\frac{1}{2}x}$; therefore the limits of x are 0 and ∞,

$$\text{and } d\theta = -\tfrac{1}{2} \frac{e^{-\frac{1}{2}x}}{\sqrt{(1-e^{-x})}} \, dx = -\tfrac{1}{2} \frac{dx}{\sqrt{(e^x - 1)}};$$

$$\therefore \int_0^\infty \frac{x \, dx}{\sqrt{(e^x - 1)}} = 2\pi \log 2 \,\ldots\ldots\, (3).$$

Cor. 3. In this last integral, if we expand the denominator

$$(\varepsilon^x - 1)^{-\frac{1}{2}} = \varepsilon^{-\frac{x}{2}} + \frac{1}{2} \varepsilon^{-\frac{3x}{2}} + \frac{1\,.\,3}{1\,.\,2} \frac{1}{2^2} \varepsilon^{-\frac{5x}{2}} + \&c.$$

$$\text{and as } \int_0^\infty \varepsilon^{-mx} x \, dx = \frac{1}{m^2}, \text{ we find}$$

$$\frac{\pi}{2} \log 2 = 1 + \frac{1}{1} \cdot \frac{1}{2} \frac{1}{3^2} + \frac{1.3}{1.2} \frac{1}{2^2} \frac{1}{5^2} + \frac{1.3.5}{1.2.3} \frac{1}{2^3} \frac{1}{7^2} + \&c.$$

R. L. E.

XI.—MATHEMATICAL NOTES.

The properties of Laplace's functions, of which demonstrations are given at p. 192 of this volume, may be extended to all symmetrical homogeneous functions of two variables.

Let u be a homogeneous function of n dimensions in x and y. Then, if it be symmetrical, we have

$$u = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right).$$

But by a known theorem

$$x\frac{du}{dx} + y\frac{du}{dy} = nu;$$

and when $y = x$, we have, in consequence of the symmetry,

$$2x\frac{du}{dx} - nu = 0 \ldots\ldots\ldots (1).$$

Again, if we suppose u to be expanded in series by powers of each of the variables

$$u = \Sigma \,.\, Q_i x^{n-i} \,.\, y^i = \Sigma \,.\, Q_i y^{n-i} \,.\, x^i,$$

from the symmetry. Substituting in the preceding equation, this gives

$$\Sigma \,.\, (2i - n)\, Q_i = 0 \ldots\ldots (2).$$

σ.

2. To show that

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n) = \frac{x^{n-r}}{a^{n-r}}\left(\frac{d}{dx}\right)^n (\varepsilon^{ax}x^r).$$

Expanding the differential on the left-hand side by the theorem of Leibnitz, we have

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n) = \left(d^r + rd^{r-1}d' + \frac{r(r-1)}{1.2}\, d^{r-2}d'^2 + \&c.\right)\left(\frac{\varepsilon^{ax}x^n}{dx^r}\right),$$

where d refers to ε^{ax} and d' to x^n. Hence

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n)$$

$$= \left(a^r x^n + ra^{r-1}nx^{n-1} + \frac{r(r-1)}{1.2}\, a^{r-2}n(n-1)x^{n-2} + \&c.\right)\varepsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}}\left(a^n x^r + na^{n-1}rx^{r-1} + \frac{n(n-1)}{1.2}\, a^{n-2}r(r-1)x^{r-2} + \&c.\right)\varepsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}}\left(d^n + nd^{n-1}d' + \frac{n(n-1)}{1.2}\, d^{n-2}d'^2 + \&c.\right)\frac{(\varepsilon^{ax}x^r)}{dx^n},$$

$$= \frac{x^{n-r}}{a^{n-r}}\left(\frac{d}{dx}\right)^n (\varepsilon^{ax}x^r),$$

by the theorem of Leibnitz. This theorem is true when one or both of the quantities n and r are fractional, so that we thus arrive at the curious result, that a differential to a fractional index may be expressed by means of a differential to an integer index of a function of the same form.

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END OF VOL. II.

Fig 1.

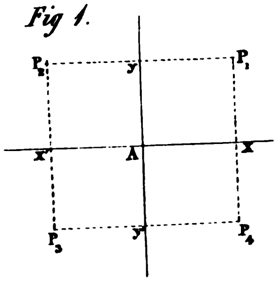

Fig 3.

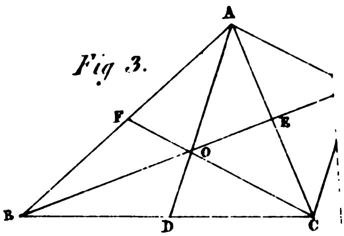

Fig 5.

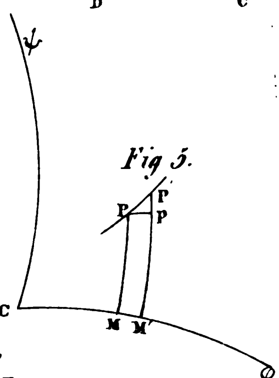

Fig 6.

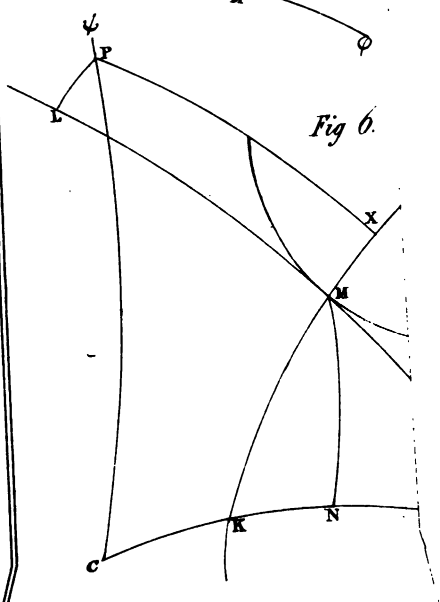

Fig. 4.

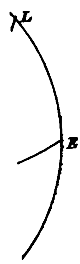

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Fig. 1

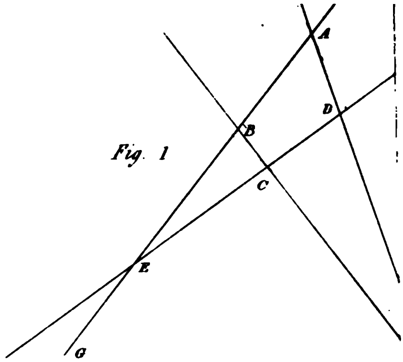


Fig. 3

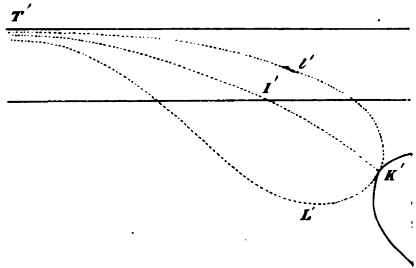
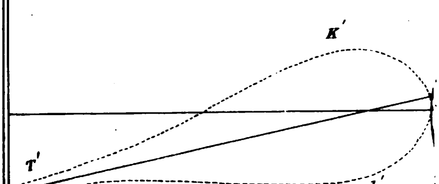


Fig. 4



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